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Problem solutions – compulsory term paper 2 in ECON3120/4120 Mathematics 2, spring 2007

Problem 1

$f(x)$ is defined if and only if $\ln(x+2)$ is defined, i.e. if and only if $x+2>0$. Thus the domain of definition is $D_f = (-2, \infty)$.

The derivative of f is

$$f'(x) = \frac{1}{2} - \frac{1}{2}x + \frac{5}{x+2} = \frac{-x^2-x+12}{2(x+2)}\,.$$

The second derivative is

$$f''(x) = -\frac{1}{2} - \frac{5}{(x+2)^2}\,.$$

(b) Every extreme point of f must be a stationary point for f. We have

$$f'(x) = 0 \iff -x^2 - x + 12 = 0 \iff x = 3 \text{ or } x = -4. \qquad (*)$$

The value $x=-4$ is outside the domain of f, so the only stationary point is $x=3$. It follows from $(*)$ that $-x^2-x+12 = -(x+4)(x-3)$, so

$$f'(x) = \frac{-(x+4)(x-3)}{2(x+2)}\,.$$

(If you don't remember how to factor quadratic polynomials, take a look at the section on quadratic equations in your textbook.) For all x in the domain of f we have $x+4>0$ and $x+2>0$, so $f'(x)$ has the same sign as $-(x-3) = 3-x$. It follows that f is (strictly) increasing in $(-2,3]$ and (strictly) decreasing in $[3,\infty)$. Hence $x=3$ is a global maximum point for $f(x)$. The maximum value is $f(3) = 5\ln 5 - 3/4 \approx 7.29719$.

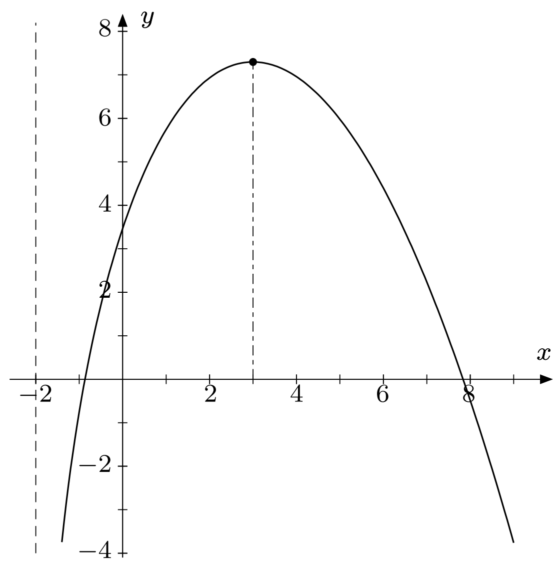
(c) The equation $f(x)=0$ has exactly two solutions. Why? The maximum value of f is $f(3)>0$. It is easy to see that $f(-1) = -3/4 < 0$, and the intermediate value theorem ("skjæringssetningen") shows that $f(x)=0$ for at least one x in $(-1,3)$. Also, $f(10) = -20 + 5\ln 12 \approx -7.575 < 0$, so there must be at least one solution of $f(x)=0$ between 3 and 10. Since f is strictly increasing in $(-2,3]$ and strictly decreasing in $[3,\infty)$, there cannot be more than one zero ("nullpunkt") of

f in either of those intervals. *Conclusion:* The equation $f(x) = 0$ has exactly two solutions. (For the curious: The two roots are $x_1 \approx -0.867547$ and $x_2 \approx 7.83515$.)

We could have managed without finding particular values of x that make $f(x)$ negative by noting that $f(x)$ tends to $-\infty$ both as x tends to -2 from the right and as x tends to ∞. It is immediately clear that $\lim_{x\to(-2)^+} f(x) = -\infty$, but $\lim_{x\to\infty} f(x)$ is a bit more tricky, since $f(x)$ is the sum of two terms that tend to ∞ and one that tends to $-\infty$. However, we can write

$$f(x) = x\Big(\frac{1}{2} - \frac{1}{4}x + \frac{5\ln(x+2)}{x}\Big). \qquad (**)$$

By l'Hôpital's rule the fraction $5\ln(x+2)/x$ tends to 0 as $x \to \infty$, and therefore the expression in the big parenthesis in $(**)$ tends to $-\infty$. The factor x outside the parenthesis tends to ∞, and therefore the whole product tends to $-\infty$.



The graph of $f(x) = \frac{1}{2}x - \frac{1}{4}x^2 + 5\ln(x+2)$.

The figure shows the graph of f. Note the vertical asymptote $x = -2$. The figure confirms that f is strictly increasing in $(-2, 3]$ and strictly decreasing in $[3, \infty)$, and also that $f(x) \to -\infty$ at either "end".

(d) By means of the substitution $u = x + 2$, $du = dx$, we get

$$\int_0^4 \Big(\frac{x}{2} - \frac{x^2}{4} + 5\ln(x+2)\Big)\,dx = \int_0^4 \Big(\frac{x}{2} - \frac{x^2}{4}\Big)\,dx + 5\int_2^6 \ln u\,du$$
$$= \Big|_0^4 \Big(\frac{x^2}{4} - \frac{x^3}{12}\Big) + 5\Big|_2^6 (u\ln u - u) = -\frac{64}{3} + 30\ln 6 - 10\ln 2 \ (\approx 25.488).$$

Problem 2

(a) With the Lagrangian $\mathcal{L} = 24x - x^2 + 16y - 2y^2 - \lambda(x^2 + 2y^2 - 44)$ we get the first-order conditions

$$24 - 2x - 2\lambda x = 0 \iff 24 - 2(1+\lambda)x = 0, \tag{1}$$
$$16 - 4y - 4\lambda y = 0 \iff 16 - 4(1+\lambda)y = 0, \tag{2}$$

together with the constraint

$$x^2 + 2y^2 = 44\,. \tag{3}$$

From (1) and (2) we get $(1+\lambda)x = 12$ and $(1+\lambda)y = 4$. Therefore $\lambda \neq -1$ and

$$x = \frac{12}{1+\lambda}\,, \qquad y = \frac{4}{1+\lambda}\,.$$

It follows that $x = 3y$. Using this in (3), we get $11y^2 = 44$, so $y = \pm 2$. The stationary points of the Lagrangian are therefore

$$(x_1, y_1) = (6, 2) \text{ with } \lambda_1 = 1, \qquad (x_2, y_2) = (-6, -2) \text{ with } \lambda_2 = -3.$$

Inserting these values of x and y into $f(x, y) = 24x - x^2 + 16y - 2y^2$ yields

$$f(6, 2) = 132, \qquad f(-6, -2) = -220.$$

The maximum point is therefore $(6, 2)$.

(We know that there is a maximum point because f is continuous and the curve $x^2 + 2y^2 = 44$ is a closed and bounded set. Also, the gradient of $g(x, y) = x^2 + 2y^2$ is never zero along the constraint curve, and therefore every extreme point in our problem must satisfy the Lagrange conditions. See Theorem 14.3.1 in EMEA or Setning 14.3.1 in MA I.)

(b) We define the Lagrangian (the Lagrange function) $\mathcal{L}$ by

$$\mathcal{L}(x, y) = 24x - x^2 + 16y - 2y^2 - \lambda(x^2 + 2y^2 - 44).$$

The necessary Kuhn–Tucker conditions for a point (x, y) to be a solution of the new problem are: There must exist a λ such that

$$\mathcal{L}_1'(x, y) = 24 - 2x - 2\lambda x = 0, \tag{4}$$
$$\mathcal{L}_2'(x, y) = 16 - 4y - 4\lambda y = 0, \tag{5}$$
$$\lambda \geq 0, \quad \text{and} \quad \lambda = 0 \text{ if } x^2 + 2y^2 < 44. \tag{6}$$

If $\lambda = 0$, then (4) and (5) imply $x = 12$ and $y = 4$, but then $x^2 + 2y^2 = 176$, which contradicts the constraint $x^2 + 2y^2 \leq 44$. Therefore we must have $\lambda > 0$. Then (6) implies that $x^2 + 2y^2 = 44$. It follows that x, y, and λ must satisfy equations (1)–(3) in part (a). Since $\lambda_2 < 0$, the only usable solution is

$$x = x_1 = 6, \quad y = y_1 = 2, \quad \lambda = \lambda_1 = 1.$$

Problem 3

(a) Cofactor expansion along the first column yields

$$|\mathbf{A}_t| = 1\begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} - (-t)\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} + t\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = 4 - 0 + t(-4) = 4(1-t).$$

In particular, $|\mathbf{A}_2| = -4$, and therefore $|(\mathbf{A}_2)^3| = |\mathbf{A}_2|^3 = (-4)^3 = -64$.

It is also possible to find $|(\mathbf{A}_2)^3|$ by first calculating $(\mathbf{A}_2)^3$ and then taking the determinant. This is not a very good idea, since it involves a lot of work and a corresponding risk of mistakes in calculation, but if really want to do it that way, here are the matrices you will get:

$$(\mathbf{A}_2)^2 = \mathbf{A}_2\mathbf{A}_2 = \begin{pmatrix} 3 & 6 & 8 \\ -4 & 9 & 6 \\ 4 & 7 & 10 \end{pmatrix}, \quad (\mathbf{A}_2)^3 = (\mathbf{A}_2)^2\mathbf{A}_2 = \begin{pmatrix} 7 & 29 & 34 \\ -10 & 29 & 22 \\ 10 & 35 & 42 \end{pmatrix}.$$

(b) Matrix multiplication yields

$$\mathbf{A}_2\mathbf{B} = \begin{pmatrix} 1 & 0 & s-\frac{3}{2} \\ 0 & 1 & 3s-\frac{9}{2} \\ 0 & 0 & s-\frac{1}{2} \end{pmatrix}.$$

For $s = 3/2$ this matrix equals $\mathbf{I}_3$, and so for this value of s,

$$\mathbf{B} = \begin{pmatrix} -1 & 0 & 1 \\ -2 & 1/2 & 3/2 \\ 2 & -1/4 & -5/4 \end{pmatrix} = (\mathbf{A}_2)^{-1}.$$

Of course, we could equally well have looked at

$$\mathbf{B}\mathbf{A}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 2s-3 & s-\frac{1}{2} & 2s-3 \\ 0 & 0 & 1 \end{pmatrix}.$$

Problem 4

Differentiating (i.e. taking derivatives) with respect to t, and remembering that x and y are functions of t, we get

$$\begin{aligned} x^2 + t\,2x\dot{x} + \dot{y} &= 2, \\ \frac{2}{x}\dot{x} + 3\dot{y} &= \dot{x} + \frac{1}{y}\dot{y} + 1. \end{aligned}$$

At the point $(x, y, t) = (1, \frac{1}{2}, 1)$ we get

$$\begin{aligned} 1 + 2\dot{x} + \quad \dot{y} &= 2 \\ 2\dot{x} + 3\dot{y} &= \dot{x} + 2\dot{y} + 1 \end{aligned} \quad \iff \quad \begin{aligned} 2\dot{x} + \dot{y} &= 1 \\ \dot{x} + \dot{y} &= 1 \end{aligned}$$

with the solution $\dot{x} = dx/dt = 0$, $\dot{y} = dy/dt = 1$.

We could also have found these results by calculating differentials:

$$\begin{aligned} x^2\,dt + 2tx\,dx + dy &= 2\,dt \\ \frac{2}{x}\,dx + 3\,dy &= dx + \frac{1}{y}\,dy + dt \end{aligned}$$

At $(x, y, t) = (1, \frac{1}{2}, 1)$ these equations become

$$\begin{aligned} dt + 2\,dx + \quad dy &= 2\,dt \\ 2\,dx + 3\,dy &= dx + 2\,dy + dt \end{aligned} \quad \iff \quad \begin{aligned} 2\,dx + dy &= dt \\ dx + dy &= dt \end{aligned}$$

with the solution $dx = 0$, $dy = dt$, which gives $\dot{x} = 0$, $\dot{y} = 1$, as above.