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A couple of "Kuhn–Tucker problems" with solutions

From the textbook:

MA II: 1.1.2, 1.3.3(b), 1.4.3(b) = **FMEA:** 5.1.3, 5.3.3(b), 5.4.7(b)

From the exam problem collection:

Problem A

(a) Solve the nonlinear programming problem

$$\text{maximize } f(x, y, z) = x + \ln(1 + z) \quad \text{subject to } \begin{cases} x^2 + y^2 \leq 1 \\ x + y + z \leq 1 \end{cases}$$

(b) What is the approximate change in the optimal value of $f(x, y, z)$ if the second constraint is replaced by $x + y + z \leq 1.02$?

Problem B

Solve the following problem:

$$\text{minimize } f(x, y) = e^{x+y} + e^y + 2x + y \quad \text{s.t.} \quad x \geq -1, \; y \geq -1, \; x + y \geq 0$$

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Solution of problem A

(a) With the Lagrangian

$$\mathcal{L}(x, y, z) = x + \ln(1 + z) - \lambda(x^2 + y^2 - 1) - \mu(x + y + z - 1),$$

the necessary Kuhn–Tucker conditions are: There must exist numbers λ and μ such that

(1) $$\mathcal{L}'_1 = 1 - 2\lambda x - \mu = 0,$$

(2) $$\mathcal{L}'_2 = -2\lambda y - \mu = 0,$$

(3) $$\mathcal{L}'_3 = \frac{1}{1 + z} - \mu = 0,$$

(4) $$\lambda \geq 0 \quad (= 0 \text{ if } x^2 + y^2 < 1),$$

(5) $$\mu \geq 0 \quad (= 0 \text{ if } x + y + z < 1).$$

Of course, (x, y, z) must also satisfy the constraints, that is,

$$x^2 + y^2 \le 1, \tag{6}$$

$$x + y + z \le 1. \tag{7}$$

(b) For $f(x, y, z)$ to be defined, we must have $1 + z > 0$. Then (3) gives

$$\mu = \frac{1}{1+z} > 0. \tag{8}$$

From (2) we get

$$2\lambda y = -\mu < 0,$$

so $\lambda \ne 0$, and it follows that $\lambda > 0$ and $y < 0$. Moreover,

$$\lambda = -\frac{\mu}{2y} = -\frac{1}{2y(1+z)}. \tag{9}$$

Substituting the expressions from (8) and (9) into (1), we get

$$1 + \frac{x}{y(1+z)} - \frac{1}{1+z} = 0 \implies y(1+z) + x - y = 0 \implies x = -yz. \tag{10}$$

Since $\lambda > 0$ and $\mu > 0$, conditions (4), (5), (6), and (7) give

$$x + y + z = 1, \tag{11}$$

$$x^2 + y^2 = 1. \tag{12}$$

From (10) and (11),

$$-yz + y + z = 1 \iff (1-y)z = 1 - y.$$

We know that $y < 0$, so $1-y \ne 0$, and $z = (1-y)/(1-y) = 1$. Then $x = -yz = -y$, and (12) gives $2y^2 = 1$, so $y = -\frac{1}{2}\sqrt{2}$ ($y < 0$, remember) and $x = -y = \frac{1}{2}\sqrt{2}$.

Thus, the only admissible point that satisfies the necessary Kuhn–Tucker conditions is

$$(x, y, z) = (\tfrac{1}{2}\sqrt{2},\ -\tfrac{1}{2}\sqrt{2},\ 1).$$

The corresponding values of λ and μ are

$$\lambda = -\frac{1}{2y(1+z)} = \frac{1}{2\sqrt{2}} = \frac{1}{4}\sqrt{2} \qquad \text{and} \qquad \mu = \frac{1}{1+z} = \frac{1}{2}.$$

It is clear that the Lagrangian is concave. Hence, the point we have found really is a maximum point for f under the given constraints.

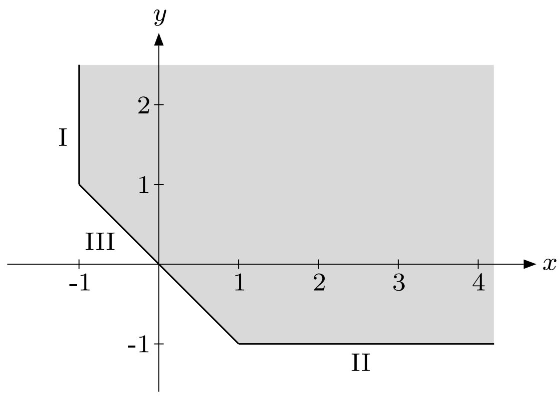
(c) If the right-hand side in the second constraint is increased by $\Delta = 0.02$, the maximum value of f is increased by approximately $\mu \cdot \Delta = \frac{1}{2} \cdot 0.02 = 0.01$.

(For the curious: The new maximum point is $(x, y, z) \approx (0.711223, -0.702967, 1.011744)$ with $\lambda \approx 0.353559$ and $\mu \approx 0.497081$.)

Problem B

We turn the problem into a maximization problem and write the constraints in "standard" form:

$$\text{maximize } -f(x,y) = -e^{x+y} - e^y - 2x - y \quad \text{subject to} \quad \begin{cases} -x \le 1 & \text{(I)} \\ -y \le 1 & \text{(II)} \\ -x - y \le 0 & \text{(III)} \end{cases}$$



The admissible set in problem B

The figure shows the admissible set. With the Lagrangian

$$L(x,y) = -e^{x+y} - e^y - 2x - y + \lambda_1 x + \lambda_2 y + \lambda_3(x+y)$$

the necessary Kuhn–Tucker conditions say that if (x, y) is to solve the problem, then there must exist numbers λ_1, λ_2, and λ_3 such that

(1) $$L'_1(x,y) = -e^{x+y} - 2 + \lambda_1 + \lambda_3 = 0$$

(2) $$L'_2(x,y) = -e^{x+y} - e^y - 1 + \lambda_2 + \lambda_3 = 0$$

(3) $$\lambda_1 \ge 0, \quad \lambda_1(-x-1) = 0$$

(4) $$\lambda_2 \ge 0, \quad \lambda_2(-y-1) = 0$$

(5) $$\lambda_3 \ge 0, \quad \lambda_3(-x-y) = 0$$

If we subtract (2) from (1) we get

(6) $$e^y - 1 + \lambda_1 - \lambda_2 = 0.$$

Now assume that $x = -1$. Because of (III) we then have $y \ge -x \ge 1$, so $-y < 1$ and therefore $\lambda_2 = 0$. Equation (6) then yields

$$\lambda_1 = 1 - e^y \le 1 - e < 0,$$

but this is impossible.

It follows that $\underline{x > -1}$, and (3) now gives $\underline{\lambda_1 = 0}$. From (1),

$$\lambda_3 = e^{x+y} + 2 - \lambda_1 = e^{x+y} + 2 > 0.$$

Therefore $x + y = 0$, and $\lambda_3 = e^0 + 2 = 3$. From (2) we get

$$e^y = -e^{x+y} - 1 + \lambda_2 + \lambda_3 = -e^0 - 1 + \lambda_2 + 3 = \lambda_2 + 1.$$

Since $\lambda_2 \geq 0$, we get $e^y \geq 1$, i.e. $y \geq 0$, and (4) yields $\lambda_2 = 0$. Hence $e^y = 1$, and $y = 0$. Since $x + y = 0$, we also get $x = 0$.

The only solution of the Kuhn–Tucker conditions is therefore

$$x = 0, \quad y = 0, \quad \lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = 0.$$

Since L is concave, $(0, 0)$ is a maximum point for $-f(x, y)$, and hence a minimum point for $f(x, y)$, over the admissible set.