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Problem solutions – compulsory term paper 1 in ECON3120/4120 Mathematics 2, spring 2007

Problem 1

(a) *Tip:* Write the function as $f(x) = \ln x + ax^{-1/2}$. Then

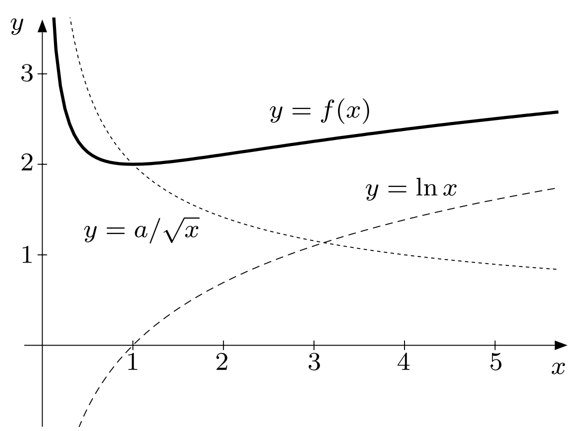
$$f'(x) = x^{-1} - \frac{1}{2}ax^{-3/2} = \frac{2\sqrt{x} - a}{2x\sqrt{x}},$$
$$f''(x) = -x^{-2} + \frac{3}{4}ax^{-5/2} = \frac{-4\sqrt{x} + 3a}{4x^2\sqrt{x}}.$$

(b) The function f is differentiable everywhere, and

$$f'(x) = 0 \iff 2\sqrt{x} = a \iff x = a^2/4.$$

Thus the only stationary point of f is $x_0 = a^2/4$. It is clear from the expression for $f'(x)$ that $f'(x) < 0$ if $0 < x < x_0$ and $f'(x) > 0$ if $x > x_0$. (The sign of $f'(x)$ is the same as the sign of $2\sqrt{x} - a$.) Thus f is strictly decreasing in $(0, x_0]$ and strictly increasing in $[x_0, \infty)$. It follows that x_0 is a global minimum point for f. There are no other extreme points.

The figure shows the graph of f together with the graphs of $\ln x$ and $a/\sqrt{x}$, with $a = 2$.



The graph of $f(x) = \ln x + a/\sqrt{x}$

Comment: Surprisingly many of you do not seem to remember the distinction between *extreme points* and *stationary points*. A stationary point for f is a point where $f'(x) = 0$, whereas an extreme point is a maximum or minimum point. If x_0 is an extreme point for f, then it must also be a stationary point for f (provided that x_0 is in the interior of the domain of f and f is differentiable at x_0). *But* a stationary point is not necessarily an extreme point, cf. the standard example $g(x) = x^3$. Here $x = 0$ is a stationary point, but it is not an extreme point, not even a local one.

If $f'(x_0) = 0$ and $f''(x_0) > 0$, then x_0 is local minimum point for f, but need not be a global minimum point. The reason is that $f'(x_0)$ and $f''(x_0)$ depend only on the local behaviour of f around x_0, not on what happens far away. Similar remarks apply to the case $f'(x_0) = 0$, $f''(x_0) < 0$.

(c) At an inflection point for f the second derivative must be zero. Now,

$$f''(x) = 0 \iff 4\sqrt{x} = 3a \iff x = (3a/4)^2 = 9a^2/16.$$

It is clear that $f''(x)$ changes sign at $x_1 = 9a^2/16$, so x_1 is an inflection point (the only one) for f. The function is convex over $(0, x_1]$ and concave over $[x_1, \infty)$.

Comment: Again, $f''(x_1) = 0$ is a necessary condition for x_1 to be an inflection point, but it is not sufficient. We also need to know that $f''(x)$ changes sign at x_0. Example: Let $g(x) = x^4$ Then $g''(0) = 0$, but g does not have an inflection point at $x = 0$. In fact g is convex over the entire real line.

(d) Writing $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \ln x + \lim\limits_{x\to 0^+} \dfrac{a}{\sqrt{x}}$ does not work, because that will lead to the undefined expression $-\infty + \infty$. (No, there is no reason to assume that this will be zero. See the warnings on page 614 in MA I or page 251 in EMEA.) But we can write

$$(*) \qquad f(x) = \frac{\sqrt{x}\ln x + a}{\sqrt{x}}\,.$$

Here, $\lim\limits_{x\to 0^+} (\sqrt{x}\ln x + a) = a > 0$, and it follows that $\lim_{x\to 0^+} f(x) = \infty$. (The denominator in $(*)$ is always positive and tends to 0.) Note that it is *not* OK to say that $\sqrt{x}\ln x + a \to a$, therefore $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} a/\sqrt{x}$. The reason is that x is supposed to tend to 0 in both the numerator and the denominator *at the same time*. We cannot first let $x \to 0$ in the numerator and afterwards let $x \to 0$ in the denominator. Just think what that would do to the limit $\lim\limits_{x\to 0} x/x$.

Since $f(x) > \ln x$, we immediately get $\lim_{x\to\infty} f(x) = \infty$. Or we could write $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \ln x + \lim\limits_{x\to\infty} \dfrac{a}{\sqrt{x}} = \infty + 0 = \infty$.

(e) The indefinite integral is

$$\int \left(\ln x + ax^{-1/2}\right) dx = x\ln x - x + 2ax^{1/2} + C = x\ln x - x + 2a\sqrt{x} + C\,.$$

Hence,

$$I_b = \Big|_b^1 (x\ln x - x + 2a\sqrt{x}\,) = 2a - 1 - b\ln b + b - 2a\sqrt{b}\,.$$

Since $b \ln b \to 0$ as $b \to 0^+$, the integral $\int_0^1 f(x)\,dx$ converges and

$$\int_0^1 \left(\ln x + a x^{-1/2}\right) dx = \lim_{b \to 0^+} I_b = 2a - 1\,.$$

Note that we cannot simply let $b = 0$ in the expression for I_b, because that would give the undefined expression $0 \ln 0$. We must examine the limit as b tends to 0 from the right.

Problem 2

(a) Matrix multiplication and subtraction yields

$$\mathbf{AB} = \begin{pmatrix} 9 & 10 & 10 \\ 13 & 15 & 14 \\ 5 & 5 & 6 \end{pmatrix}, \qquad \mathbf{A} - \mathbf{B} = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 4 \\ -1 & -1 & -1 \end{pmatrix}.$$

Cofactor expansion along the first row yields

$$|\mathbf{A} - \mathbf{B}| = 1 \begin{vmatrix} 2 & 4 \\ -1 & -1 \end{vmatrix} - 0\,|\cdots| + 2 \begin{vmatrix} -1 & 2 \\ -1 & -1 \end{vmatrix} = 2 - 0 + 6 = 8\,.$$

(There is no need to calculate the value of the cofactor corresponding to position $(1,2)$ in the matrix.)

(b) Matrix multiplication yields

$$(\mathbf{A} - \mathbf{B})\mathbf{D} = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix} = 8\,\mathbf{I}_3\,,$$

and therefore

$$(\mathbf{A} - \mathbf{B})^{-1} = \frac{1}{8}\mathbf{D}\,.$$

Comment: Many of you calculated the inverse of $\mathbf{A} - \mathbf{B}$ by the general procedure in the book, involving the cofactors etc., but in case like this, where a suggested expression for the inverse is given, such as $t\mathbf{D}$ in this problem, it is easier to check directly whether the product $(\mathbf{A} - \mathbf{B})(t\mathbf{D})$ is the identity matrix.

(c) A simple exercise in matrix algebra:

$$\begin{aligned} \mathbf{BX} = \mathbf{AX} + \mathbf{C} &\iff (\mathbf{B} - \mathbf{A})\mathbf{X} = \mathbf{C} \\ &\iff \mathbf{X} = (\mathbf{B} - \mathbf{A})^{-1}\mathbf{C} = -(\mathbf{A} - \mathbf{B})^{-1}\mathbf{C} = -\frac{1}{8}\mathbf{DC} = -\frac{1}{8}\begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix} \end{aligned}$$

Problem 3

The equation is separable, $\dot{x} = f(t)g(x)$, with $f(t) = t^3\sqrt{t^2 + 1}$ and $g(x) = e^{-x}$. Since $g(x) \neq 0$ for all x, the differential equation has no constant solutions. The standard recipe for separable equations yields

$$\int e^x\,dx = \int t^3\sqrt{t^2+1}\,dt\,.$$

With the substitution $u = \sqrt{t^2+1}$ we get $t^2+1 = u^2$, so $2t\,dt = 2u\,du$ and

$$\begin{aligned} e^x &= \int t^2\sqrt{t^2+1}\,t\,dt = \int (u^2-1)u\,u\,du = \int (u^4-u^2)\,du \\ &= \tfrac{1}{5}u^5 - \tfrac{1}{3}u^3 + C = \tfrac{1}{5}(t^2+1)^{5/2} - \tfrac{1}{3}(t^2+1)^{3/2} + C\,. \end{aligned}$$

For a solution that passes through $(t_0, x_0) = (0,0)$ we must have

$$e^0 = \tfrac{1}{5} - \tfrac{1}{3} + C \iff C = 1 - \tfrac{1}{5} + \tfrac{1}{3} = \tfrac{17}{15}\,.$$

The required solution is therefore

$$x = \ln\left(\tfrac{1}{5}(t^2+1)^{5/2} - \tfrac{1}{3}(t^2+1)^{3/2} + \tfrac{17}{15}\right).$$

Problem 4

(a) Taking differentials gives

$$\begin{aligned} e^x uv\,dx + e^x v\,du + e^x u\,dv + 2(\ln v)\,dy + \frac{2y}{v}\,dv &= 0 \\ e^u\,dy + y\,e^u\,du - v\,dx - x\,dv &= 0. \end{aligned}$$

(b) At $P = (x, y, u, v) = (0, e^2, 1, e)$ we get

$$\begin{aligned} e\,dx + e\,du + dv + 2\,dy + 2e\,dv &= 0 \\ e\,dy + e^3\,du - e\,dx &= 0 \end{aligned}$$

Rearranging yields

$$\begin{aligned} e\,du + (2e+1)\,dv &= -e\,dx - 2\,dy \\ e^3\,du \qquad\qquad &= e\,dx - e\,dy \end{aligned}$$

Solving this linear system for du and dv, we get

$$du = \frac{1}{e^2}\,dx - \frac{1}{e^2}\,dy, \qquad dv = -\frac{e^2+1}{e(2e+1)}\,dx + \frac{1-2e}{e(2e+1)}\,dy.$$

It follows that the partial derivatives of u and v at P are

$$u'_x = \frac{1}{e^2}\,, \quad u'_y = -\frac{1}{e^2}\,, \quad v'_x = -\frac{e^2+1}{e(2e+1)}\,, \quad v'_y = \frac{1-2e}{e(2e+1)}\,.$$

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(c) Moving from $(x_0, y_0) = (0, e^2)$ to $(x_1, y_1) = (0.1, e^2)$ corresponds to giving x and y increments $dx = 0.1$ and $dy = 0$, respectively. The corresponding changes in the values of u and v are then approximately

$$du = u'_x\,dx + u'_y\,dy = 0.1\,u'_x = 0.1/e^2 \approx 0.0135\,,$$
$$dv = v'_x\,dx + v'_y\,dy = 0.1\,v'_x = -0.1(e^2+1)/(2e^2+e) \approx -0.0479\,,$$

and the new values of u and v are

$$u_1 \approx 1 + du \approx 1.0135\,, \qquad v_1 \approx e + dv \approx 2.6703\,.$$

These are the approximate values given by the linear approximation to u and v around P. More accurate approximations show that $u_1 \approx 1.0132$ and $v_1 \approx 2.6690$, correct to 4 decimal places.

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