ECON 4120 Seminar 7

30, 54, 86, 96, 110, 132, 138(a).

$\boxed{30}$

(a) The Lagrangian function is

$$\mathcal{L}(x,y) = x^2 + y^2 - 2x + 1 - \lambda\left(\tfrac{1}{4}x^2 + y^2 - b\right)$$

With first order conditions

, 2 cases

(1) $\mathcal{L}_x = 2x - 2 - \frac{1}{2}x\lambda = 0$

(2) $\mathcal{L}_y = 2y - 2y\lambda = 0 \Rightarrow \lambda = 1$ or $y = 0$

$\underline{\lambda = 1}$: Then from (1): $2x - 2 - \frac{1}{2}x = 0 \iff \frac{3}{2}x = 2 \iff x = \frac{4}{3}$

The constraint gives $\frac{1}{4}\left(\frac{4}{3}\right)^2 + y^2 = b$

$$\Updownarrow$$

$$y^2 = b - \frac{4}{9}$$

$$\Updownarrow$$

$$y = \pm\sqrt{b - 4/9}$$

Thus ~~one~~ two stationary points are ~~is~~ $(x,y) = \left(\frac{4}{3}, \pm\sqrt{b - 4/9}\right)$

$\boxed{30}$

(a)

Consider the case when $y=0$, by (2).

Then, the constraint gives

$$\frac{1}{4}x^2 + 0 = b \iff x = \pm\sqrt{4b}, \quad \boxed{(x,y) = (\pm\sqrt{4b}, 0)}$$

We have four stationary point. Calculating the value of the objective function at each point will reveal maxima and minima points;

$$\left(\frac{4}{3}\right)^2 + \sqrt{b-4/9}^{\,2} - 2\left(\frac{4}{3}\right) + 1 = \frac{16}{9} + b - 4/9 - \frac{8}{3} + 1 = \underline{\underline{b - \frac{3}{9}}}$$

$$\left(\frac{4}{3}\right)^2 + \left(-\sqrt{b-4/9}\right)^2 - 2\left(\frac{4}{3}\right) + 1 = \underline{\underline{b - \frac{3}{9}}} \rightarrow \text{minimum}$$

$$\sqrt{4b}^{\,2} + 0 - 2\sqrt{4b} + 1 = \underline{\underline{4b - 4\sqrt{b} + 1}}$$

$$\left(-\sqrt{4b}\right)^2 + 0 + 2\sqrt{4b} + 1 = \underline{\underline{4b + 4\sqrt{b} + 1}} \rightarrow \text{maximum}$$

maximum is at $(-\sqrt{4b}, 0)$

minimum is at $\left(\frac{4}{3}, \sqrt{b-4/9}\right)$ and $\left(\frac{4}{3}, -\sqrt{b-4/9}\right)$.

$\left(\text{show that } 4b - 4\sqrt{b} + 1 > b - \frac{3}{9} \text{ for } b > \frac{4}{9}\right)$

30

(b)

The maximum value is

$$f^*(b) = 4b + 4\sqrt{b} + 1$$

$$\frac{d f^*(b)}{db} = 4 + \frac{4}{2\sqrt{b}} = 4 + \frac{2}{\sqrt{b}}$$

Condition (1) gives, for $x = -\sqrt{4b}$:

$$2(-\sqrt{4b}) - 2 - \frac{1}{2}(-\sqrt{4b})\lambda = 0$$

Solving for λ:

$$-\frac{1}{2}(-\sqrt{4b})\lambda = 2 + 2\sqrt{4b}$$

$$\lambda = \frac{4}{\sqrt{4b}} + 4$$

$$\lambda = \frac{2}{\sqrt{b}} + 4 = \frac{d f^*(b)}{db}$$

54

$$f(x,y) = x^2 - y^2 - xy - x^3$$

Calculate first and second order partial derivatives.

$f'_x = 2x - y - 3x^2$, $f'_y = -2y - x$

$f''_{xx} = 2 - \cancel{2}6x$, $f''_{yy} = -2$

$f''_{xy} = -1$

Stationary points must satisfy:

$$2x - y - 3x^2 = 0 \qquad (1)$$

$$-2y - x = 0 \qquad (2)$$

(2) gives $x = -2y$. Plug this into (1):

$2(-2y) - y - 3(-2y)^2 = 0 \iff -5y - 12y^2 = 0$

$\iff -12y\left(y + \frac{5}{12}\right) = 0$

$\iff y = -\frac{5}{12}$ or $y = 0$

It follows that $f(x,y)$ has stationary points $(0,0)$ and $\left(\frac{5}{6}, -\frac{5}{12}\right)$

$\boxed{54}$

Use the second derivative test to characterize the stationary points as maximum, minimum or saddle points:

$A = f''_{xx}$, $B = f''_{xy}$, $C = f''_{yy}$.

(x,y)	A	B	C	$AC - B^2$	max/min/saddle?
$(0,0)$	2	-1	-2	-5	Saddle point
$\left(\frac{5}{6}, -\frac{5}{12}\right)$	-3	-1	-2	5	local maximum.

86

(a)

$$f(x,y,z) = x^2 + x + y^2 + z^2$$

The Lagrangian is

$$\mathcal{L}(x,y,z) = x^2 + x + y^2 + z^2 - \lambda(x^2 + 2y^2 + 2z^2 - 16)$$

(the boundary of a spheroid)

The constraint defines a closed and bounded set ~~[illegible]~~ so the extreme value theorem says that we have both maximum and minimum points.

Extreme points must satisfy the constraint and the first order conditions:

(1) $\mathcal{L}_x = 2x + 1 - 2x\lambda = 0$

(2) $\mathcal{L}_y = 2y - 4y\lambda = 0 \iff 2(1-2\lambda)y = 0$

(3) $\mathcal{L}_z = 2z - 4z\lambda = 0 \iff 2(1-2\lambda)z = 0$

(2) and (3) gives two cases to consider:

A: If $\lambda = 1/2$, (2) and (3) say nothing about y and z. From (1), $x = -1$. Inserting this into the constraint:

$2y^2 + 2z^2 = 16 - (-1)^2 = 15$. Hence, all points $(-1, y, z)$ where $2y^2 + 2z^2 = 15 \iff y^2 + z^2 = \frac{15}{2}$ satisfy (1)-(3) and the constraint.

86

(a)

B: if $\lambda \neq \frac{1}{2}$, then (2) and (3) imply $y = z = 0$.

The constraint gives $x^2 + 0 + 0 = 16$

$$x = \pm 4$$

So $(4,0,0)$ and $(-4,0,0)$ satisfy (1)-(3) and the constraint. To find the maxima and minima, find the value of $f(x,y,z)$ at each point:

$f(-1, y, z) = 1 - 1 + y^2 + z^2 = \frac{15}{2}$ → minima ~~minimum~~

$f(4,0,0) = 16 + 4 = 20$ → maximum

$f(-4,0,0) = 16 - 4 = 12$

Note that there are infinitely many minimum points, the ones where $x = -1$ and $y^2 + z^2 = \frac{15}{2}$.

There is one maximum point, at $(x,y,z) = (4,0,0)$.

186

(b)

An extreme point for ~~f(x~~ $f(x,y,z)$ over the set S must lie on the boundary of S, which is defined by the constraint in (a), or in the interior of S, where $x^2 + 2y^2 + 2z^2 < 16$

In the latter case, it must be a stationary point for $f(x,y,z)$, which must satisfy

$$\left.\begin{aligned} f'_x &= 2x + 1 = 0 \\ f'_y &= 2y = 0 \\ f'_z &= 2z = 0 \end{aligned}\right\} \Rightarrow (x,y,z) = \left(-\tfrac{1}{2}, 0, 0\right)$$

Check the value of $f(x,y,z)$ at this point:

$$f\left(-\tfrac{1}{2}, 0, 0\right) = \frac{1}{4} - \frac{1}{2} + 0 + 0 = \underline{\underline{-\frac{1}{4}}},$$

Which is clearly a minimum.

Then $(x,y,z) = (4,0,0)$ must be a maximum for f over S.

$\boxed{96}$

(a) Differentiate:

$$y \cdot 1 + xy' + 2yy' + 2 + 2y' = 0 \qquad (*)$$

$$(x + 2y + 2)y' = -2 - y$$

$$y' = -\frac{2+y}{2+x+2y}$$

Differentiate $(*)$ wrt x:

$$y' + y' + xy'' + 2(y')^2 + 2yy'' + 2y'' = 0$$

$$\Leftrightarrow \quad (x + 2y + 2)y'' = -2y' - 2(y')^2$$

$$(x + 2y + 2)y'' = \frac{2(2+y)}{2+x+2y} - 2\left(\frac{2+y}{2+x+2y}\right)^2$$

$$y'' = \frac{2(2+y)(2+x+2y) - 2(2+y)^2}{(2+x+2y)^3}$$

$$y'' = \frac{2(2+y)(2+x+2y-2-y)}{(2+x+2y)^3}$$

$$y'' = \frac{2(2+y)(x+y)}{(2+x+2y)^3}$$

96

(10)

(b) The Lagrangian is

$$\mathcal{L}(x,y) = xy + y^2 + 2x + 2y - \lambda(6x + 10y - m)$$

The necessary conditions for maximum are

(1) $\mathcal{L}'_x = y + 2 - 6\lambda = 0$

(2) $\mathcal{L}'_y = x + 2y + 2 - 10\lambda = 0$

(3) $6x + 10y = m$

(1) and (2) imply $\lambda = \dfrac{y+2}{6} = \dfrac{x + 2y + 2}{10}$

$$\Updownarrow$$

$$10y + 20 = 6x + 12y + 12$$

$$\Updownarrow$$

$$8 = 6x + 2y \qquad (4)$$

Solve (3) and (4):

$\rightarrow$ y^*: $\quad 8 - 2y + 10y = m \iff 8y = m - 8 \iff \underline{\underline{y^* = \frac{1}{8}m - 1}}$

$\rightarrow$ x^*: $\quad 8 = 6x + 2\left(\frac{1}{8}m - 1\right) \iff 6x = 10 - \frac{1}{4}m \iff \underline{\underline{x^* = \frac{5}{3} - \frac{1}{24}m}}$

$\rightarrow$ $\lambda = \dfrac{y^* + 2}{6} = \dfrac{\frac{1}{8}m - 1 + 2}{6} = \underline{\underline{\frac{1}{48}m + \frac{1}{6}}}$

96

(11)

(b) Do these values for x^* and y^* really solve the maximization problem?

Transform the problem into a one-variable problem:

$$6x + 10y = m \iff y = \frac{m}{10} - \frac{3}{5}x$$

$$U(x, y(x)) \equiv \hat{U}(x)$$

$$= x\left(\frac{m}{10} - \frac{3}{5}x\right) + \left(\frac{m}{10} - \frac{3}{5}x\right)^2 + 2x$$

$$+ 2\left(\frac{m}{10} - \frac{3}{5}x\right)$$

$$= -\frac{6}{25}x^2 + \left(\frac{4}{5} - \frac{1}{50}m\right)x + \frac{m^2}{100} + \frac{m}{5}$$

With x in $\left(0, \frac{m}{6}\right)$ (for $y > 0$).

Check that $\hat{U}'(x^*) = 0$ and $x^* = \frac{5}{3} - \frac{1}{24}m$ is an interior point in $\left(0, \frac{m}{6}\right)$.

Also note that $\hat{U}$ is concave. Thus x^* maximizes $\hat{U}$, and so also x^* and y^* maximize U.

96

(c)

$$U(x^*, y^*) = \left(\frac{5}{3} - \frac{1}{24}m\right)\left(\frac{1}{8}m - 1\right) + \left(\frac{1}{8}m - 1\right)^2$$

$$+ \; 2\left(\frac{5}{3} - \frac{1}{24}m\right) + 2\left(\frac{1}{8}m - 1\right)$$

$$= \frac{5}{24}m - \frac{5}{3} - \frac{1}{8 \cdot 24}m^2 + \frac{1}{24}m + \frac{1}{64}m^2 - \frac{2}{8}m + 1$$

$$+ \; \frac{10}{3} - \frac{2}{24}m + \frac{2}{8}m - 2$$

$$U(x^*, y^*) = \frac{1}{96}m^2 + \frac{1}{6}m + \frac{2}{3}$$

$$\frac{dU(x^*, y^*)}{dm} = \frac{1}{48}m + \frac{1}{6} = \lambda(m).$$

When $m = 20$, $\dfrac{dU(x^*, y^*)}{dm} = \dfrac{20}{48} + \dfrac{1}{6} = \underline{\underline{\dfrac{7}{12}}}$

(d) Since consumption cannot be negative, we cannot have a solution with $x < 0$ or $y < 0$. If $m \leq 8$, $y^* = \frac{m}{8} - 1 \leq 0$ and $x^* = \frac{5}{3} - \frac{m}{24} \geq \frac{m}{6}$ (exceeds the budget).

Optimal $\underline{\underline{x = \frac{m}{6}}}$, optimal $\underline{\underline{y = 0}}$.

If $m \geq 40$, optimal $\underline{\underline{x = 0}}$, optimal $\underline{\underline{y = \frac{m}{10}}}$ (the whole budget is spent on y. Corner solution)

$\boxed{11c}$

(a) The Lagrangian is

$$\mathcal{L}(x,y,z) = x^2 + y^2 + z - \lambda(x^2 + 2y^2 + 4z^2 - 1)$$

And the necessary conditions

$$\mathcal{L}'_x = 2x - 2\lambda x = 0 \quad (1)$$
$$\mathcal{L}'_y = 2y - 4\lambda y = 0 \quad (2)$$
$$\mathcal{L}_z = 1 - 8\lambda z = 0 \quad (3)$$
$$x^2 + 2y^2 + 4z^2 = 1 \quad (4)$$

(1) gives $2x(1-\lambda) = 0$, which gives two cases:

A: If $x=0$, we get, from (2), $2y(1-2\lambda) = 0$, which implies $y=0$ or $\lambda = \frac{1}{2}$.

A1 if $y=0$, (4) implies $4z^2 = 1 - x^2 - 2y^2 = 1 \iff z^2 = \frac{1}{4}$ and $z = \pm\frac{1}{2}$. This gives two candidate points:

P1: $(0,0,\frac{1}{2})$ with $\lambda = \frac{1}{4}$ $\quad f(0,0,\frac{1}{2}) = \frac{1}{2}$

P2: $(0,0,-\frac{1}{2})$ with $\lambda = -\frac{1}{4}$ $\quad f(0,0,-\frac{1}{2}) = -\frac{1}{2}$

16

(a)

A2: if $\lambda = \frac{1}{2}$, then (3) yields $1 - 8 \cdot \frac{1}{2} \cdot z = 0$

$\Updownarrow$

X is still 0, so (4) implies $z = \frac{1}{4}$

$$0 + 2y^2 + 4\frac{1}{4^2} = 1$$

$$\Updownarrow$$

$$2y^2 = \frac{3}{4} \iff y = \pm\sqrt{\frac{3}{8}}$$

This gives two new candidate points:

P3: $(0, \sqrt{3/8}, 1/4)$ with $\lambda = \frac{1}{2}$, $f(0, \sqrt{3/8}, 1/4) = \frac{5}{8}$

P4: $(0, -\sqrt{3/8}, 1/4)$ with $\lambda = \frac{1}{2}$, $f(0, -\sqrt{3/8}, 1/4) = \frac{5}{8}$

B: If $\lambda = 1$, then (3) implies $z = \frac{1}{8}$, and (2) gives $y = 0$. From (4), we get

$$x^2 + 0 + 4 \cdot \frac{1}{64} = 1 \iff x^2 = \frac{15}{16} \iff x = \pm\sqrt{\frac{15}{16}}$$

$$x = \pm\frac{\sqrt{15}}{4}$$

Candidate points:

P5: $(\sqrt{15}/4, 0, 1/8)$, $\lambda = 1$, $f(\sqrt{15}/4, 0, 1/8) = \frac{17}{16}$

P6: $(-\sqrt{15}/4, 0, 1/8)$, $\lambda = 1$, $f(-\sqrt{15}/4, 0, 1/8) = \frac{17}{6}$

$\boxed{11c}$ (15)

(a) Points P1 – P6 exhaust all possibilities for extreme points. Looking at function values for each point, it is clear that f attains its maximum value at P5 and P6 with $f_{max} = \frac{17}{16}$. and its minimum value at P2 with $f_{min} = -\frac{1}{2}$.

(Note that (4) defines a closed and bounded set, so f has a max and min over that set, due to the EVT (extreme value theorem).)

(b) Using formula 14.2.3 in EMEA (and MA1)

$$\Delta f^* = f^*(1 + 0.02) - f^*(1) \approx \lambda \cdot dc = 1 \cdot 0.02 = 0.02.$$

where dc is a small change in the constraint, and f^* is a function $f^*(c)$ of the constraint.

$\boxed{132}$

$$f(x,y) = \frac{1}{2}x^2 - x + ay(x-1) - \frac{1}{3}y^3 + a^2y^2 \quad , \quad \forall (x,y)$$

FOCs:

(1) $f'_x = x - 1 + ay = 0 \iff x = 1 - ay$

(2) $f'_y = a(x-1) - y^2 + 2a^2 y = 0$

Use $x = 1 - ay$ in (2):

$$a(1 - ay - 1) - y^2 + 2a^2 y = 0$$

$$-a^2 y - y^2 + 2a^2 y = 0$$

$$y(a^2 - y) = 0$$

This implies $y = 0$ or $y = a^2$

Which implies $x = 1$ or $x = 1 - a^3$.

Stationary points at $(1,0)$ and $(1-a^3, a^2)$.

Second derivative test:

$A = f''_{xx} = 1$, $B = f''_{xy} = a$, $C = f''_{yy} = -2y + 2a^2$

$AC - B^2 = 2a^2 - 2y - a^2 = a^2 - 2y$

~~For $(1,0)$, $a^2 - 2y = a^2 > 0$~~

~~For $(1-a^3, a^2)$, $a^2 - 2y = -a^2 < 0 \Rightarrow$ Saddle point~~

132

If $a \neq 0$:

$(x,y) = (1,0) \Rightarrow AC - B^2 = a^2 > 0$

$(x,y) = (1-a^3, a^2) \Rightarrow AC - B^2 = -a^2 < 0$

$(1,0)$ is a ~~minimum~~ local minimum, because $f''_{xx} > 0$.

$(1-a^3, a^2)$ is a saddle point.

If $a = 0$:

Second derivative test is inconclusive for local extrema:
However, ~~by~~ when $a=0$, we get

$f(x,y) = \frac{1}{2}x^2 - x - \frac{1}{3}y^3$

$f(1,0) = \frac{1}{2} - 1 - 0 = -\frac{1}{2}$

If y were to increase by an arbitrarily small amount, Δy, $f(1, \Delta y)$ would be $< -\frac{1}{2}$. If y decreased by a small amount, $-\Delta y$, $f(1, -\Delta y)$ would be $> -\frac{1}{2}$, so this cannot be a local max or min, and must be a saddle point.

$\boxed{138}$

(a)

Two constraints, two Lagrange multipliers:

$$\mathcal{L}(x,y,z) = e^x + y + z - \lambda(x+y+z-1) - \delta(x^2+y^2+z^2-1)$$

FOC:

$$\mathcal{L}'_x = e^x - \lambda - 2\delta x = 0 \qquad (1)$$

$$\mathcal{L}'_y = 1 - \lambda - 2\delta y = 0 \qquad (2)$$

$$\mathcal{L}'_z = 1 - \lambda - 2\delta z = 0 \qquad (3)$$

$$x + y + z = 1 \qquad (4)$$

$$x^2 + y^2 + z^2 = 1 \qquad (5)$$

(2) and (3) imply $2\delta y = 2\delta z$, which gives two cases $y = z$ and $\delta = 0$.

<u>A: If $y = z$</u>, the constraints (4) and (5) yield:

$x + 2y = 1$ and $x^2 + 2y^2 = 1$

$\hookrightarrow x = 1 - 2y$.

$\hookrightarrow (1-2y)^2 + 2y^2 = 1 \iff 1 - 4y + 4y^2 + 2y^2 = 1$

$\iff 6y^2 - 4y = 0$

$\boxed{138}$

(a)

$\Longleftrightarrow \quad y(6y-4)=0$

Hence $y=0$ or $y=\frac{2}{3}$.

Remembering that $y=z$, and $x=1-2y$, this gives two candidate points:

$P1: (1,0,0)$, with $\lambda=1$, $\delta=\frac{1}{2}(e-1)$, $f(1,0,0)=e$

$P2: (-1/3, 2/3, 2/3)$, with $\lambda=\frac{1}{3}+\frac{2}{3}e^{-1/3}$, $\delta=\frac{1}{2}-\frac{1}{2}e^{-1/3}$, $f(-1/3, 2/3, 2/3)=e^{-1/3}+4/3$.

B: If $\delta=0$, (2) yields $\lambda=1$, (1) gives $e^{x}=1 \Leftrightarrow x=0$

(4) and (5) give $y+z=1$ and $y^2+z^2=1$,

hence $y=1-z \Rightarrow (1-z)^2+z^2=1$

$\hookrightarrow 1-2z+z^2+z^2=1$

$\hookrightarrow z(2z-2)=0 \Rightarrow z=0$ or $z=1$.

This gives candidates:

$P3: (0,0,1)$, with $\lambda=1$, $\delta=0$, $f(0,0,1)=2$

$P4: (0,1,0)$, with $\lambda=1$, $\delta=0$, $f(0,1,0)=2$

138

(a)

Note that $e^{-1/3} + 4/3 < 1 + 4/3 < e$,

and $e > 2$, so $(1,0,0)$ is a maximum point for $f(x,y,z)$. Also, check that $e^{-1/3} + 4/3 > 2$, so that $(0,1,0)$ and $(0,0,1)$ are minimum points.