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ECON3120/4120/4140/4145 Mathematics 2/3: An application of Kuhn–Tucker

The following is an information economics application of the Kuhn–Tucker conditions. This note is common to Mathematics 2 and Mathematics 3, and it is expected that at least Mathematics 2 students will find problem 2 demanding.

Assumptions for both problems

For both problems below, let $p \in (0, 1)$ and $k > 0$ be constants, and U and V given strictly increasing, strictly concave C^2 (utility) functions defined on $[0, \infty)$. We assume that U' and V' take all positive values (i.e.: both tend to infinity at zero and to zero at infinity), that $U(0) = V(0) = 0$, and that

$$U'(x) > V'(x), \qquad \text{all } x.$$

Problem 1

It can be shown – and in problem 2 you shall do precisely that – that the maximization of problem 2 reduces to

$$\max_{q,Q} p[U(Q) - U(q) + V(q)] + (1-p)V(q) - k(pQ + (1-p)q) \quad \text{subject to } Q \geq q. \qquad (*)$$

- Assume that the problem has a solution. Solve in terms of the derivatives of the utility functions U and V.
- The interpretations are that U' and V' are marginal utilities, and k is marginal production cost. Comments?

Problem 2

Let

$$f(t,q,T,Q) = pT + (1-p)t - k(pQ + (1-p)q)$$

(interpretation: you have two kinds of customers, you offer them the menu of *either buy the quantity q and pay t, or buy the quantity Q and pay T*. There is a fraction p of one type («U», named after its utility function) and this has by assumption the highest willingness-to-pay – if we then think of Q as the bigger quantity, then this is reserved for this higher-demand customer. Not knowing which customer is which, you will with probability p meet one which accepts the Q offer, and with probability $1-p$ one which accepts the other. Production cost is k per unit. f is then the expected profit.)

Consider the maximization

$$\max_{(t,q),(T,Q)} f(t,q,T,Q) \quad \text{subject to}$$

$$U(Q) - T \geq 0 \tag{1}$$

(Interpretation: the «U» type customer must break-even in order to accept the offer.)

$$V(q) - t \geq 0 \tag{2}$$

(Interpretation: the «V» type customer must break-even in order to accept the offer.)

$$U(Q) - T \geq U(q) - t \tag{3}$$

(Interpretation: if the «U» type customer shall accept the «pay T for Q» deal, it must be preferred to the «pay t for q deal. (If this is violated, then the probabilities in the definition of f will be wrong, invalidating the model.))

$$V(q) - t \geq V(Q) - T \tag{4}$$

(Interpretation: if the «V» type customer shall accept the «pay t for q» deal, it must be preferred to the «pay T for Q deal. Why bother which customer type chooses what deal? Because otherwise the probabilities p and $1-p$ are wrong.)

Problem 2 is about establishing that – under the above assumptions on U, V, k, p – this problem reduces to (*) of problem 1. Proceed as follows:

- Show that (1) or (2) must be active:
  Assume for contradiction that they are both inactive. Then both t and T can be increased by the same small amount x without violating (1), (2); show that this does not affect (3), (4). What happens to f?
- Recall that $U' > V'$, with $U(0) = V(0)$. Use this to show that the RHS of (3) is greater than 0 if (2) holds; hence the LHS of (3) is ≥ 0, hence (1) holds *automatically* – and can therefore be dropped.
- Use the two previous bullet points to show that (2) must be active.

- Eliminating t by inserting for (2), yields

$$\max_{q,(T,Q)} pT + (1-p)V(q) - k(pQ + (1-p)q) \quad \text{subject to}$$

$$U(Q) - T \geq U(q) - V(q) \tag{5}$$

$$0 \geq V(Q) - T \tag{6}$$

Write down the Lagrangian (recall which way the inequalities are in our standard Kuhn-Tucker problem!) and use one of the first-order conditions to show that (5) is active (i.e.: can it not be? What happens if the respective Lagrange multiplier is zero?)

- With (5) being active, we can eliminate $T = U(Q) - U(q) + V(q)$:

$$\max_{q,Q} p[U(Q) - U(q) + V(q)] + (1-p)V(q) - k(pQ + (1-p)q) \quad \text{subject to}$$

$$U(Q) - V(Q) - (U(q) - V(q)) \geq 0 \tag{7}$$

- Now employ the following trick: Put $\Delta = U - V$. Then the left hand side of (7) equals

$$\Delta(Q) - \Delta(q) = \int_q^Q \Delta(x)\mathrm{d}x = \int_q^Q (U'(x) - V'(x))\mathrm{d}x. \tag{8}$$

Show that (8) is nonnegative – i.e. (7) holds – if and only if $Q \geq q$.

(It is actually possible to transform one of these problems into a concave one and hence apply sufficient conditions.)