The problems posted in «Messages» for the March 9 seminar

The following note *sketches* the solution.

Problem 1 Let $Q(\mathbf{x}) = \mathbf{x}'\mathbf{A}\mathbf{x}$. By the extreme value theorem, both m and M exist and are attained somewhere on the unit sphere.

We have for any $\mathbf{y} \neq 0$ that $\mathbf{y}'\mathbf{A}\mathbf{y} = ||\mathbf{y}||^2 Q(\mathbf{x})$ where $\mathbf{x} = \mathbf{y}/||\mathbf{y}||$ has length 1. Thus $Q(\mathbf{y})$ has the same sign as $Q(\mathbf{x})$, and if $m > 0$ this is positive; if $M < 0$ it is negative; if $m = 0$, it is nonnegative, and 0 is attained; if $M = 0$, it is nonpositive, and 0 is attained; if $M > 0 > m$ we have both signs attained.

Consider now the problem to max/min $Q(\mathbf{x})$ over the unit sphere – and rewrite the constraint into $||\mathbf{x}||^2 = 1$. The Lagrange condition is $(\mathbf{A} + \mathbf{A}')\mathbf{x} = 2\lambda\mathbf{x}$, and since $\mathbf{A}$ is assumed symmetric, it says precisely that the solution must be an eigenvector. To get the possible values, left-multiply by $\mathbf{x}/2$ to get $Q(\mathbf{x}) = \lambda||\mathbf{x}||^2$ i.e. $= \lambda$. Thus the possible max/min values are the eigenvalues, and we pick the largest for M and the smallest for m.

Problem 2

- Under the assumption, $\mathbf{AV} = \mathbf{VD}$, and by inspection, column i on each side says $\mathbf{A}\mathbf{v}^{(i)} = d_{ii}\mathbf{v}^{(i)}$ (since $\mathbf{D}$ is diagonal), so these are eigenvalues and eigenvectors. Since V is by assumption invertible, the columns are linearly independent, which shows the existence of n linearly independent eigenvectors.

- $\mathbf{V}$ satisfies the assumption of the previous bullet point, and $\mathbf{C}$ the ones that point imposed on $\mathbf{D}$, so the conclusions for $\mathbf{A}$ carry over except with c_{ii} in place of d_{ii}. To prove the last claim, form the product $\mathbf{B}^p$ which equals $\mathbf{V}\mathbf{C}^p\mathbf{V}^{-1}$, and $\mathbf{C}^p = \mathbf{D}$.

- If $\mathbf{V}' = \mathbf{V}^{-1}$ then $\mathbf{V}$ takes the form $\mathbf{VCV}'$, with transpose $\mathbf{V}''\mathbf{C}'\mathbf{V}'$, which equals $\mathbf{B}$ since $\mathbf{C}$ is symmetric. Now the eigenvalues of $\mathbf{B}$ have the same sign as the eigenvalues of $\mathbf{A}$, and so to show that $\mathbf{B}$ is positive (semi-)definite if $\mathbf{A}$ is, we only need to point out that (semi-)definiteness is defined for $\mathbf{B}$, i.e. if it is symmetric.

- Without loss of generality, scale all eigenvectors $\mathbf{v}^{(i)}$ to $||\mathbf{v}^{(i)}|| = 1$. Let $\mathbf{u}$ and $\mathbf{w}$ be eigenvectors with respective eigenvalues μ and λ. Since $\mathbf{u}'\mathbf{A}\mathbf{w} = \mathbf{w}'\mathbf{A}'\mathbf{u}$ always and $\mathbf{A}$ is symmetric, we calculate $\mathbf{u}'\mathbf{A}\mathbf{w} = \mathbf{u}'\lambda\mathbf{w}$ and $\mathbf{w}'\mathbf{A}\mathbf{u} = \mathbf{w}'\mu\mathbf{u}$. So $\lambda\mathbf{u}\cdot\mathbf{w} = \mu\mathbf{u}\cdot\mathbf{w}$, which if $\mu \neq \lambda$ implies orthogonality.
  Now recall that element (i, j) of a matrix product, is row i of the left dot column j of the right. And row i of $\mathbf{V}'$ equals column j of $\mathbf{V}$, so the element is the dot product of columns i and j. On the main diagonal, we get 1 by the assumed scaling, and off diagonal, the product is zero because of the orthogonality for different eigenvalues.