Answer to “Lecture question” in Lecture 5

In Lecture 5 under the heading **Final equation**, we studied the Gaussian VAR

\[
\begin{pmatrix}
Y_t \\
X_t
\end{pmatrix}
= \begin{pmatrix}
\pi_{11} & \pi_{12} \\
\pi_{21} & \pi_{22}
\end{pmatrix}
\begin{pmatrix}
Y_{t-1} \\
X_{t-1}
\end{pmatrix} + \begin{pmatrix}
\varepsilon_{yt} \\
\varepsilon_{xt}
\end{pmatrix},
\] (1)

where \( \begin{pmatrix}
\pi_{11} & \pi_{12} \\
\pi_{21} & \pi_{22}
\end{pmatrix} \) is the matrix of autoregressive coefficients and we assume that

\[
\begin{pmatrix}
\varepsilon_{yt} \\
\varepsilon_{xt}
\end{pmatrix} \sim IID \left(0, \begin{pmatrix}
\sigma_{yt}^2 & \sigma_{yx}^2 \\
\sigma_{xy}^2 & \sigma_{xt}^2
\end{pmatrix} \right) \forall t
\] (2)

As an exercise, you were asked to show that (1) can be reduced to the so called **final equation** for \( Y_{t+1} \)

\[
Y_{t+1} = (\pi_{11} + \pi_{22})Y_t + (\pi_{12}\pi_{21} - \pi_{22}\pi_{11})Y_t + \varepsilon_{yt+1} - \pi_{22}\varepsilon_{yt} + \pi_{12}\varepsilon_{xt} \equiv \varepsilon_t.
\] (3)

Try by repeated substitution: The answer is on the next page!
Answer:
Solve the equation in the first row in (1) for $X_{t-1}$:

$$X_{t-1} = \left(1/\pi_{12}\right)Y_t - \left(\pi_{11}/\pi_{12}\right)Y_{t-1} - \left(1/\pi_{12}\right)\varepsilon_{yt}. \tag{4}$$

Substitution in the second row of (1) gives

$$X_t = \frac{\pi_{22}}{\pi_{12}}Y_t + \left(\pi_{21} - \frac{\pi_{11}}{\pi_{12}}\right)Y_{t-1} - \frac{\pi_{22}}{\pi_{12}}\varepsilon_{yt} + \varepsilon_{xt}, \pi_{12} \neq 0. \tag{5}$$

Finally: Change $t$ to $t+1$ in the first row of (1), and replace $X_t$ by the expression on the right hand side of (5):

$$Y_{t+1} = \pi_{11}Y_t + \pi_{12}\left(\frac{\pi_{22}}{\pi_{12}}Y_t + \left(\pi_{21} - \frac{\pi_{11}}{\pi_{12}}\right)Y_{t-1} - \frac{\pi_{22}}{\pi_{12}}\varepsilon_{yt} + \varepsilon_{xt}\right) + \varepsilon_{yt+1} \tag{6}$$

Collecting terms gives

$$Y_{t+1} = (\pi_{11} + \pi_{22})Y_t + (\pi_{12}\pi_{21} - \pi_{22}\pi_{11})Y_{t-1} + \varepsilon_{yt+1} - \pi_{22}\varepsilon_{yt} + \pi_{12}\varepsilon_{xt}, \tag{7}$$

which is what we should find. Clearly this equation must hold for all periods, so we can write

$$Y_t = (\pi_{11} + \pi_{22})Y_{t-1} + (\pi_{12}\pi_{21} - \pi_{22}\pi_{11})Y_{t-2} + \varepsilon_{yt} - \pi_{22}\varepsilon_{yt-1} + \pi_{12}\varepsilon_{xt-1}, \tag{8}$$

the final equation for $Y_t$ defined by the system (1).

A final equation expresses an endogenous variable by the “its own lags” and exogenous random variables. No lags of other endogenous variables are allowed in a final equation expression.

Additional questions, for review:

1. Write down the homogenous difference equation that corresonds to the final equation for $Y_t$.

2. Write down the associated characteristic equation. How does the characteristic roots of the homogenous equation relate to the conditions for dynamic stability and covariance stationarity of $Y_t$?

3. Derive the final equation for $X_t$. What are the requirements for stationarity of $X_t$.

4. What are the conditions for stationarity of the vector time-series $y_t = (Y_t, X_t)'$?