Lecture note 6

GIVE and 2SLS equivalence.

The equivalence of \( \beta_{GIV} \) and \( \beta_{2SLS} \) can be shown “both ways”. DM page 323-324 has a short argument showing the equivalence by starting from \( \beta_{2SLS} \).

In the slide set to Lecture 6 we take the opposite direction. This brief note fills in a couple of steps. We start by GIVE:

\[
\hat{\beta}_{1,GIV} = (\hat{W}_1'X_1)^{-1}\hat{W}_1'y_1. \tag{1}
\]

with

\[
\hat{W}_1 = \begin{pmatrix} Z_1 & \hat{Y}_1 \end{pmatrix}, \tag{2}
\]

\[
X_1 = \begin{pmatrix} Z_1 & Y_1 \end{pmatrix} \tag{3}
\]

and

\[
\hat{Y}_1 = P_{W_1}Y_1. \tag{4}
\]

where the matrices are given in the slide set, in particular:

\[
P_{W_1} = W_1(W_1'W_1)^{-1}W_1'. \tag{5}
\]

Using the partitioning, the product \( \hat{W}_1'X_1 \) becomes:

\[
\hat{W}_1'X_1 = \begin{pmatrix} Z_1 & \hat{Y}_1 \end{pmatrix}' \begin{pmatrix} Z_1 & Y_1 \end{pmatrix} = \begin{pmatrix} Z_1'Z_1 & Z_1'Y_1 \\ \hat{Y}_1'Z_1 & \hat{Y}_1'Y_1 \end{pmatrix}
\]

so

\[
\hat{\beta}_{1,GIV} = \begin{pmatrix} Z_1'Z_1 & Z_1'Y_1 \\ \hat{Y}_1'Z_1 & \hat{Y}_1'Y_1 \end{pmatrix}^{-1} \begin{pmatrix} Z_1'y_1 \\ \hat{Y}_1'y_1 \end{pmatrix}
\]

We now focus on \( Z_1'Y_1 \) and \( \hat{Y}_1'Y_1 \). Start with the second:

\[
\hat{Y}_1'Y_1 = (P_{W_1}Y_1)'Y_1
\]

The reduced form residual \( e_1 \) matrix is given by

\[
e_1 = Y_1 - P_{W_1}Y_1 = M_{W_1}Y_1
\]

where

\[
M_{W_1} = I - P_{W_1}
\]

Using this we get

\[
(P_{W_1}Y_1)'Y_1 = (P_{W_1}Y_1)'(\hat{Y}_1 + e_1) = Y_1'P_{W_1}\hat{Y}_1 = \hat{Y}_1'\hat{Y}_1
\]

\[
= Y_1'P_{W_1}\hat{Y}_1 = \hat{Y}_1'\hat{Y}_1
\]
since \((P_W, Y_1)'e_1 = Y_1'(P_W'(I - P_W)Y_1 = 0.1\)

Next, consider \(Z_1'Y_1\). We want to show that \(\hat{Y}_1'Z_1 = Z_1'Y_1\). We can start by writing \(\hat{Y}_1'Z_1\):

\[
\hat{Y}_1'Z_1 = (Y_1' - e'_1)Z_1 = Y_1'Z_1 - e'_1Z_1
\]

Here \(e'_1Z_1 = 0\) (since the effect of the exogenous \(Z_1\) has been regressed out, but try to show formally if you want). But then

\[
\hat{Y}_1'Z_1 = Y_1'Z_1 \Leftrightarrow Z_1'\hat{Y}_1 = Z_1'Y_1 \quad (7)
\]

Replacing \(\hat{Y}_1'Y_1\) by \(\hat{Y}_1'\hat{Y}_1\) and \(Z_1'Y_1\) by \(Z_1'\hat{Y}_1\) we get

\[
\hat{\beta}_{1, GIV} = \left( \begin{array}{cc} Z_1'Z_1 & Z_1'\hat{Y}_1 \\ \hat{Y}_1'Z_1 & \hat{Y}_1'\hat{Y}_1 \end{array} \right)^{-1} \left( \begin{array}{c} \hat{Y}_1'y_1 \\ Z_1'y_1 \end{array} \right)
\]

which is the expression for \(\hat{\beta}_{1, 2SLS}\).

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1Remember:

\[
P_{W_1}' = P_{W_1}
\]

\[
P_{W_1}'P_{W_1} = [W_1(W_1'W_1)^{-1}W_1'][W_1(W_1'W_1)^{-1}W_1'] = P_{W_1}
\]