Note 1 to Computer class: The natural logarithm

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August 22, 2011

1 Logarithms and rates of change

We often make use of the approximation
\[
\ln \left(1 + \frac{\Delta Y}{Y}\right) = \ln \left(\frac{Y + \Delta Y}{Y}\right) = \ln (Y + \Delta Y) - \ln Y \\
\approx \frac{\Delta Y}{Y}, \quad \frac{\Delta Y}{Y} \text{ is small.}
\]

We will see how this approximation works, and how good it is.

When we consider marginal changes we have the precise relationship, i.e., from the derivative of a function
\[
\frac{d\ln Y}{dY} = \frac{1}{Y}, \quad \text{and hence } d\ln Y = \frac{dY}{Y}.
\]

To get to the approximation we need the following properties of the logarithmic function: \(\ln X\):
\[
\ln X = \begin{cases} 
= (X - 1), & X = 1 \\
< (X - 1), & X \neq 1
\end{cases}
\]

The relationships holds because the function \(f (X) = (X - 1)\) is a straight line with slope coefficient \(\frac{d(X-1)}{dX} = 1\), while the function \(g (X) = \ln X\) is concave, with a slope coefficient that decreases with increasing \(X\):
\[
\frac{d\ln X}{dX} = \frac{1}{X} \begin{cases} < 1, & X > 1 \\
> 1, & X < 1
\end{cases}
\]

This result is illustrated in figure 1.

If we let \(X = \frac{Y + \Delta Y}{Y} = 1 + \frac{\Delta Y}{Y}\), we get the relationship between logarithms and rates of change:
\[
\ln \left(1 + \frac{\Delta Y}{Y}\right) = \ln \left(\frac{Y + \Delta Y}{Y}\right) = \ln (Y + \Delta Y) - \ln Y \leq \frac{\Delta Y}{Y}.
\]

Changes in logarithms will therefore never give larger values than the exact rates of change. But how good is the approximation, and why does it become poorer when

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*This note is a translation of Appendix 3.A in Bårdsen and Nymoen (2011).*
the rate of changes becomes larger? We can investigate this by taking a second order Taylor expansion of $\ln X$ around the value $X_0 = 1$:

$$\ln X \approx \ln X_0 + \frac{1}{X_0} (X - X_0) - \left( \frac{1}{X_0^2} \right) \frac{(X - X_0)^2}{2}$$

$$= \ln 1 + 1 (X - 1) - \frac{1}{2} (X - 1)^2$$

$$\ln X \approx (X - 1) - \frac{1}{2} (X - 1)^2. \quad (1)$$

If we substitute $X = \frac{Y + \Delta Y}{Y}$, as above, we can see how close the approximation is, depending on the size of the rate of change:

$$\ln \left( \frac{Y + \Delta Y}{Y} \right) = \ln (Y + \Delta Y) - \ln Y \approx \frac{\Delta Y}{Y} - \frac{1}{2} \left( \frac{\Delta Y}{Y} \right)^2.$$ 

Clearly the approximation is best for smallish rates of change, but it also works well for a 10% change in $Y$, as the table shows:

<table>
<thead>
<tr>
<th>$\frac{\Delta Y}{Y}$</th>
<th>0.01</th>
<th>0.05</th>
<th>0.1</th>
<th>0.2</th>
<th>0.5</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ln \left( 1 + \frac{\Delta Y}{Y} \right)$</td>
<td>0.00995</td>
<td>0.04879</td>
<td>0.09531</td>
<td>0.18232</td>
<td>0.40547</td>
<td>0.69315</td>
</tr>
</tbody>
</table>
The standard error of a log-linear model estimated by OLS

Assume that we have estimated a model for $\ln Y_i$, so that we can write:

$$\ln Y_i = \ln \hat{Y}_i + \hat{\varepsilon}_i,$$

where $\hat{Y}_i = e^{\hat{\beta}_0 + X_i \hat{\beta}_1}$ in the case of a single explanatory variable. The relationship for the variable $Y_i$ becomes:

$$Y_i = \hat{Y}_i e^{\hat{\varepsilon}_i}.$$

From (1), and by dropping the second order term for simplicity, we have:

$$\ln Y_i - \ln \hat{Y}_i = \hat{\varepsilon}_i \approx \frac{Y_i - \hat{Y}_i}{Y_i}, \quad (2)$$

Equation (2) shows that the residual $\hat{\varepsilon}_i$ has an interpretation as a relative prediction error. Hence $100\hat{\varepsilon}_i \approx 100 \left( \frac{Y_i - \hat{Y}_i}{Y_i} \right)$ can be interpreted as percentage prediction error. The standard errors of the regression:

$$100\hat{\sigma}_e = 100 \sqrt{\frac{1}{n - 2} \sum_{i=1}^{n} \left( \frac{Y_i - \hat{Y}_i}{Y_i} \right)^2}$$

$$\approx 100 \sqrt{\frac{1}{n - 2} \sum_{i=1}^{n} \left( \frac{Y_i - \hat{Y}_i}{Y_i} \right)^2}$$

is the percentage unexplained standard deviation in the dependent variable. For example, if we have $\hat{\sigma}_e = 0.01$, it means that 1% of the standard deviation of the dependent variable is unexplained by the model we have estimated.

This interpretation is independent of the number of explanatory variables. If we have $k$ variables, we replace $n - 2$ by $n - k - 1$.

References