Exam in: ECON 4160: Econometrics: Modelling and Systems Estimation—Answer notes to evaluators.

Day of exam: 12 January 2022

Time of day: 09:00—14:00

This is a 5 hour home exam.

Guidelines:
In the grading, question A gets 60 % and B 40 %.

Question A (60 %)

Consider the deterministic dynamic model with two endogenous variables, Q_t and P_t:

$$Q_t = a_{q0} + a_{qp}P_t + a_{qq}Q_{t-1} \tag{1}$$
$$P_t = b_{p0} + b_{pq}Q_t + b_{pp}P_{t-1} \tag{2}$$

1. Show that the stationary solution, if it exists, is defined by the equation system:

$$\begin{pmatrix} (1-a_{qq}) & -a_{qp} \\ -b_{pq} & (1-b_{pp}) \end{pmatrix} \begin{pmatrix} Q^* \\ P^* \end{pmatrix} = \begin{pmatrix} a_{q0} \\ b_{p0} \end{pmatrix} \tag{3}$$

where Q^* and P^* denote the stationary solutions. A: If a stationary solution exists, it implies that $Q_t = Q_{t-1} = Q^*$ and $P_t = P_{t-1} = P^*$. Substitute in (1)-(2):

$$Q^* = a_{q0} + a_{qp}P^* + a_{qq}Q^*$$
$$P^* = b_{p0} + b_{pq}Q^* + b_{pp}P^*$$

Rearrange to

$$(1-a_{qq})Q^* - a_{qp}P^* = a_{q0}$$
$$-b_{pq}Q^* + (1-b_{pp})P^* = b_{p0}$$

and use matrix notation to give (3).

2. Show that the stationary solution can be expressed as:

$$\begin{pmatrix} Q^* \\ P^* \end{pmatrix} = \begin{pmatrix} \frac{(1-b_{pp})}{c} & \frac{a_{qp}}{c} \\ \frac{b_{pq}}{c} & \frac{(1-a_{qq})}{c} \end{pmatrix} \begin{pmatrix} a_{q0} \\ b_{p0} \end{pmatrix} \tag{4}$$

where c is given as

$$c = (1 - a_{qq} - b_{pp} - a_{qp}b_{pq} + a_{qq}b_{pp}). \tag{5}$$

A: Solve the model with respect to Q^* and P^* Can for example invert the coefficient matrix:

$$\begin{pmatrix} (1-a_{qq}) & -a_{qp} \\ -b_{pq} & (1-b_{pp}) \end{pmatrix}^{-1} = \frac{1}{1-a_{qq}-b_{pp}-a_{qp}b_{pq}+a_{qq}b_{pp}} \begin{pmatrix} (1-b_{pp}) & a_{qp} \\ b_{pq} & (1-a_{qq}) \end{pmatrix}$$

$$c = (1 - a_{qq} - b_{pp} - a_{pq}b_{pq} + a_{qq}b_{pp})$$

3. Show that the reduced form of (1)-(2) can be written as:

$$\begin{pmatrix} Q_t \\ P_t \end{pmatrix} = \begin{pmatrix} -\frac{1}{a_{qp}b_{pq}-1} & -\frac{1}{a_{qp}b_{pq}-1}a_{qp} \\ -\frac{1}{a_{qp}b_{pq}-1}b_{pq} & -\frac{1}{a_{qp}b_{pq}-1} \end{pmatrix} \begin{pmatrix} a_{q0} \\ b_{p0} \end{pmatrix} + \Phi \begin{pmatrix} Q_{t-1} \\ P_{t-1} \end{pmatrix}, \tag{6}$$

where the matrix $\mathbf{\Phi}$ is given as:

$$\mathbf{\Phi} = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix} = \begin{pmatrix} -\frac{1}{a_{qp}b_{pq}-1}a_{qq} & -\frac{1}{a_{qp}b_{pq}-1}a_{qp}b_{pp} \\ -\frac{1}{a_{qp}b_{qp}-1}b_{pq}a_{qq} & -\frac{1}{a_{pq}b_{pq}-1}b_{pp} \end{pmatrix}. \tag{7}$$

For reference, denote the vector with constants in (6) by $\mathbf{\Upsilon}$, i.e.:

$$\mathbf{\Upsilon} = \begin{pmatrix} -\frac{1}{a_{qp}b_{pq}-1} & -\frac{1}{a_{qp}b_{pq}-1}a_{qp} \\ -\frac{1}{a_{qp}b_{pq}-1}b_{pq} & -\frac{1}{a_{qp}b_{pq}-1} \end{pmatrix} \begin{pmatrix} a_{q0} \\ b_{p0} \end{pmatrix} \tag{8}$$

A: In matrix notation (1)-(2) becomes

$$\begin{pmatrix} 1 & -a_{qp} \\ -b_{pq} & 1 \end{pmatrix} \begin{pmatrix} Q_t \\ P_t \end{pmatrix} = \begin{pmatrix} a_{q0} \\ b_{p0} \end{pmatrix} + \begin{pmatrix} a_{qq} & 0 \\ 0 & b_{pp} \end{pmatrix} \begin{pmatrix} Q_{t-1} \\ P_{t-1} \end{pmatrix}$$

and the reduced form becomes:

$$\begin{pmatrix} Q_t \\ P_t \end{pmatrix} = \begin{pmatrix} 1 & -a_{qp} \\ -b_{pq} & 1 \end{pmatrix}^{-1} \begin{pmatrix} a_{q0} \\ b_{p0} \end{pmatrix} + \begin{pmatrix} 1 & -a_{qp} \\ -b_{pq} & 1 \end{pmatrix}^{-1} \begin{pmatrix} a_{qq} & 0 \\ 0 & b_{pp} \end{pmatrix} \begin{pmatrix} Q_{t-1} \\ P_{t-1} \end{pmatrix}$$

$$\begin{pmatrix} 1 & -a_{qp} \\ -b_{pq} & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{1}{a_{qp}b_{pq}-1} & -\frac{1}{a_{qp}b_{pq}-1}a_{qp} \\ -\frac{1}{a_{qp}b_{pq}-1}b_{pq} & -\frac{1}{a_{qp}b_{pq}-1} \end{pmatrix}$$

The matrix Φ is:

$$\begin{aligned} \Phi &= \begin{pmatrix} -\frac{1}{a_{qp}b_{pq}-1} & -\frac{1}{a_{qp}b_{pq}-1}a_{qp} \\ -\frac{1}{a_{qp}b_{pq}-1}b_{pq} & -\frac{1}{a_{qp}b_{pq}-1} \end{pmatrix} \begin{pmatrix} a_{qq} & 0 \\ 0 & b_{pp} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{a_{qp}b_{pq}-1}a_{qq} & -\frac{1}{a_{qp}b_{pq}-1}a_{qp}b_{pp} \\ -\frac{1}{a_{qp}b_{pq}-1}b_{pq}a_{qq} & -\frac{1}{a_{qp}b_{pq}-1}b_{pp} \end{pmatrix} \end{aligned}$$

4. What is the condition on the eigenvalues of $\mathbf{\Phi}$ (characteristic roots) that secure global asymptotic stability of the model given by (1) and (2)? A: The two eigenvalues must be less than one in magnitude.

5. Consider the stochastic model:

$$Q_t = a_{q0} + a_{qp}P_t + a_{qq}Q_{t-1} + e_{qt}, \tag{9}$$
$$P_t = b_{p0} + b_{pq}Q_t + b_{pp}P_{t-1} + e_{pt}, \tag{10}$$

where e_{qt} and e_{pt} are two gaussian white-noise disturbances:

$$\begin{pmatrix} e_{qt} \\ e_{pt} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \omega_1^2 & \omega_{12} \\ \omega_{12} & \omega_2^2 \end{pmatrix}\right) \text{ for all } t. \tag{11}$$

What is the condition for weak stationarity of the time series variables defined by this model? A (9)-(10) has a reduced form where the disturbance term are linear combinations of the SEM disturbances, hence they are stationary. The stationarity condition is therefore a generalization of the stability condition: It is that neither of the two eigenvalues of $\mathbf{\Phi}$ are equal to one in magnitude (no unit-roots).

6. Assume weak stationarity. What is the expression for the expectation of the time series Q_t? A: Given stationarity, we can take expectation on both sides of (9) and (10). This gives a system with the same mathematical structure as the stationary system above. Therefore:

$$E(Q_t) = Q^* = \frac{(1 - b_{pp})}{c}a_{q0} + \frac{a_{qp}}{c}b_{p0}$$

7. The reduced form of (9)-(10) can be written as the VAR:

$$\boldsymbol{y}_t = \boldsymbol{\Phi} \boldsymbol{y}_{t-1} + \boldsymbol{\Upsilon} + \boldsymbol{\varepsilon}_t \tag{12}$$

where $\boldsymbol{y}_t$ is the $2x1$ vector with Q_t and P_t as elements, and $\boldsymbol{\varepsilon}_t$ is a $2x1$ vector with the gaussian white noise time series ε_{qt} and ε_{pt} as elements:

$$\boldsymbol{\varepsilon}_t = \begin{pmatrix} \varepsilon_{qt} \\ \varepsilon_{pt} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} \right) \text{ for all } t. \tag{13}$$

(a) Explain why, in general, $\sigma_{12} \neq 0$.
A: The two VAR error-terms will be correlated even in the case of $\omega_{12} = 0$, because of the simultaneous determination of Q_t and P_t.

(b) Assume that $\omega_{12} = 0$ in (11). Give an algebraic expression for σ_{12} for this special case.
A:

$$\begin{aligned}
\begin{pmatrix} \varepsilon_{qt} \\ \varepsilon_{pt} \end{pmatrix} &= \begin{pmatrix} -\frac{1}{a_{qp}b_{pq}-1} & -\frac{1}{a_{qp}b_{pq}-1}a_{qp} \\ -\frac{1}{a_{qp}b_{pq}-1}b_{pq} & -\frac{1}{a_{qp}b_{pq}-1} \end{pmatrix} \begin{pmatrix} e_{qt} \\ e_{pt} \end{pmatrix} \\
&= \begin{pmatrix} -\frac{1}{a_{qp}b_{pq}-1}e_{qt} - \frac{a_{qp}}{a_{qp}b_{pq}-1}e_{pt} \\ -\frac{1}{a_{qp}b_{pq}-1}e_{pt} - \frac{b_{pq}}{a_{qp}b_{pq}-1}e_{qt} \end{pmatrix} \\
&= \frac{1}{1 - a_{qp}b_{pq}} \begin{pmatrix} e_{qt} + a_{qp}e_{pt} \\ e_{pt} + b_{pq}e_{qt} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\sigma_{12} = E(\varepsilon_{qt}\varepsilon_{pt}) &= (\frac{1}{a_{pq}b_{pq} - 1})^2 E((-e_{qt} - a_{qp}e_{pt}) \cdot (-e_{pt} - b_{pq}e_{qt})) \\
&= (\frac{1}{a_{qp}b_{pq} - 1})^2 E((-e_{qt} - a_{qp}e_{pt}) \cdot (-e_{pt} - b_{pq}e_{qt}))
\end{aligned}$$

$$\begin{aligned}
E((-e_{qt} - a_{pq}e_{pt}) \times (-e_{pt} - b_{pq}e_{qt})) &= E(e_{qt}e_{pt} + a_{pq}e_{pt}e_{pt} + e_{qt}b_{pq}e_{qt} + a_{pq}e_{pt}b_{pq}e_{qt})) \\
&= a_{qp}\omega_2^2 + b_{pq}\omega_1^2
\end{aligned}$$

$$\sigma_{12} = (\frac{1}{a_{pq}b_{pq} - 1})^2 (a_{pq}\omega_2^2 + b_{pq}\omega_1^2)$$

8. Discuss the identification of (9) and (10) under the assumption that all the coefficients of the model equations are different from zero.
A: As there the covariance matrix of the SEM disturbances is unrestricted the order and rank conditions can be used to discuss identification. (9) and (10) excludes one variable each. Hence, they are both identified on the (necessary) order condition. Since all the coefficient of the two model equations are non-zero, it is implied that the sufficient rank condition is also satisfied.

Question B (40 %)

Assume that the Data Generating Process (DGP) is given by (12) and (13) in Question A and that the DGP is stationary . Consider the conditional model equation of Q_t:

$$Q_t = \phi_0 + \phi_1 Q_{t-1} + \beta_0 P_t + \beta_1 P_{t-1} + \epsilon_t \tag{14}$$

Assume that you have time series observations of Q_t and P_t, $t = 1, 2, ..., T$.

1. Explain why ϵ_t and ε_{pt} are uncorrelated.
   A: This is due to valid conditioning on P_t. Since the only way that ε_{pt} can influence Q_t is through P_t the remainder ϵ_t must be uncorrelated with ε_{pt}.

2. Explain why the OLS estimator $\hat{\beta}_0$ has probability limit:

$$\text{plim } (\hat{\beta}_0) = \frac{\sigma_{12}}{\sigma_2} \tag{15}$$

   A: The VAR (12) and (13) is gaussian. (14) is the correct conditional model of Q_t given P_t. In the conditional expectation function the parameter of P_t is therefore the regression coefficient $\frac{\sigma_{12}}{\sigma_2}$. As OLS is a consistent estimator of the parameters of the conditional expectation function (15) the probability limit in (15) is implied.

3. Assume that the DGP given by (12) and (13) is non stationary, and that Q_t and P_t are two cointegrated I(1) variables.

   (a) Explain how you can estimate the coefficients of the cointegration relationship by the estimation of a conditional model
   A: Between two I(1) variables, there can logically be only one cointegration relationship. Cointegration also implies equilibrium-correction. Hence, if we re-parameterize (14) as an unrestricted ECM model for ΔQ_t we estimate the long-run slope coefficient of the cointegration relationship as th ratio of the coefficients of the Q_t and P_t. Clarifying to include the relevant algebraic expressions in the answer.

   (b) Explain the condition under which P_t is weakly exogenous with respect to the estimation of the coefficients of the cointegration relationship, and when it is not weakly exogenous.
   A: The ECM-term should be insignificant in the marginal model for ΔP_t.

4. Assume that the specification of the DGP is unknown but that we can assume that the time series Q_t and P_t are either $I(1)$ or $I(0)$. We do not know if there is a long-run relationship between them.

   (a) Use the results in Table 1 to decide whether the time series variables are $I(1)$ or $I(0)$. Explain your reasoning.
   A: The reasonable and good answer her is to not reject null of unit-root. for both. can choose t-adf with no or low degree of augmentation.

   (b) Table 2 shows estimation results that can be used to the null hypothesis of no long run relationship between Q_t and P_t. Give you conclusion and explain your reasoning.
   (In the table, DQ and DP denote the first differences of the two time series.)
   A: -1.74 is not significant when compared to relevant critical value in Table 3 (-3.21 for example).

Tables with estimation results and facimile of table with critical values for ECM-test

Table 1: Dickey-Fuller tests of unit-root in Q_t and P_t.

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Unit-root tests
 The sample is: 4 - 201 (201 observations and 2 variables)

Q: ADF tests (T=198, Constant+Trend; 5%=-3.43 1%=-4.01)
D-lag    t-adf       beta Y_1    sigma   t-DY_lag  t-prob
 2     -3.110       0.92740  0.08692    0.4023  0.6879
 1     -3.090       0.92853  0.08673     1.634  0.1039
 0     -2.911       0.93284  0.08710

P: ADF tests (T=198, Constant+Trend; 5%=-3.43 1%=-4.01)
Dlag    t-adf       beta Y_1    sigma   t-DY_lag  t-prob
 2     -3.494*      0.90517   0.1023     1.116  0.2660
 1     -3.376       0.90909   0.1024    0.1970  0.8440
 0     -3.388       0.90979   0.1021
```

Table 2: Regression of DQ_t on DQ_{t-1}, DP_t, DP_{t-1}, Q_{t-1}, P_{t-1} and $Constant$.

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EQ(1) Modelling DQ by OLS
      The estimation sample is: 3 - 201

                 Coefficient  Std.Error  t-value  t-prob
DQ_1                0.117010    0.07153     1.64  0.1035
Constant          -0.0250875    0.02816   -0.891  0.3741
DP                 0.0758406    0.06093     1.24  0.2147
DP_1              -0.0184715    0.06120   -0.302  0.7631
Q_1               -0.0308654    0.01774    -1.74  0.0834
P_1                0.0305891    0.02175     1.41  0.1613

sigma              0.0884937  RSS                1.51140855
R^2                   0.0333  F(5,193) =       1.33 [0.253]
Adj.R^2           0.00825591  log-likelihood        203.217
no. of observations      199  no. of parameters           6
mean(DQ)         -0.00276963  se(DQ)              0.0888613

AR 1-2 test:      F(2,191)   =  0.52789 [0.5907]
ARCH 1-1 test:    F(1,197)   = 0.027473 [0.8685]
Normality test:   Chi^2(2)   = 0.043112 [0.9787]
Hetero test:      F(10,188) =  0.67027 [0.7511]
Hetero-X test:    F(20,178) =  0.78892 [0.7247]
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Table 3: Facsimile from article by Ericsson and MacKinnon.

Table 3. Response surface estimates for critical values of the ECM test of cointegration $\kappa_c(k)$: with a constant term.

k	Size (%)	θ_∞	(s.e.)	θ_1	θ_2	θ_3	$\hat{\sigma}$
1	1	−3.4307	(0.0006)	−6.52	−4.7	−10	0.00790
	5	−2.8617	(0.0003)	−2.81	−3.2	37	0.00431
	10	−2.5668	(0.0003)	−1.56	2.1	−29	0.00332
2	1	−3.7948	(0.0006)	−7.87	−3.6	−28	0.00847
	5	−3.2145	(0.0003)	−3.21	−2.0	17	0.00438
	10	−2.9083	(0.0002)	−1.55	1.9	−25	0.00338
3	1	−4.0947	(0.0005)	−8.59	−2.0	−65	0.00857
	5	−3.5057	(0.0003)	−3.27	1.1	−34	0.00462
	10	−3.1924	(0.0002)	−1.23	2.1	−39	0.00364
	1	−4.3555	(0.0006)	−8.90	−6.7	−31	0.00959