Exam in: ECON 4160: Econometrics: Modelling and Systems Estimation

Day of exam: 11 December 2023

Time of day: 09:00—14:00

This is a 5 hour home exam.

Guidelines:
In the grading, question A gets 40 %, B 60 %.

Question A (40 %)

Consider the ADL-model equation:

$$Y_t = \phi_0 + \phi_1 Y_{t-1} + \beta_0 X_t + \beta_1 X_{t-1} + \epsilon_t, t = 1, 2, ..., T, \quad (1)$$

where X_t is a stationary time series which is integrated of order zero, $X_t \sim I(0)$. ϵ_t denotes an error-term which is white-noise and linearly uncorrelated with X_t, X_{t-1}, Y_{t-1}, hence $E(\epsilon_t \mid X_t, X_{t-1}, Y_{t-1}) = 0$.

1. Explain how the following model equations can be obtained as special cases of (1):
   (a) AR model.
   (b) Distributed lag model.
   (c) Static model.
   (d) Model in differences.

   Answer note for evaluators: The models are defined by coefficient restrictions. (Chapter 6.5 of DEEMM, and Lecture 5). AR(1): $\beta_0 = \beta_1 = 0$; DL : $\phi_1 = 0$; Static: $\beta_1 = \phi_1 = 0$; Differences: $\beta_1 = -\beta_0$ and $\phi_1 = 1$.

2. Show how (1) can be written by the use of lag-operator notation.
   Answer note for evaluators:

$$(1 - \phi_1 L)Y_t = \phi_0 + (\beta_0 + \beta_1 L)X_t + \epsilon_t$$

$$\pi(L)Y_t = \phi_0 + \beta(L)X_t + \epsilon_t$$

   where $\pi(L) = 1 - \phi_1 L$ and $\beta(L) = \beta_0 + \beta_1 L$. (Chapter 3.5.1 and 6.3 in DEEMM, Lecture 2 and 5). Of course, other symbols can be used to denote the two lag polynomials.

3. Assume coefficients $\phi_1 = 0.5$, $\beta_0 = 0.8$, $\beta_1 = -0.3$ in (1). Calculate the impact multiplier and the three first dynamic multipliers of the dependent variable with respect to a one-period unit-change in the explanatory variable (an impulse of $+1$ to X).
   Answer note for evaluators:

$$\delta_0 = 0.8$$
$$\delta_1 = -0.3 + 0.5 \times 0.8 = 0.1$$
$$\delta_2 = 0.5 \times 0.1 = 0.05$$
$$\delta_3 = 0.5 \times 0.05 = 0.025$$

   Ch 6.3 of DEEMM and Lecture 5.

4. (a) Explain what we mean by a long-run multiplier of Y with respect to X.
   (b) Show that for equation (1) and the coefficients given in question A3, the long-run multiplier is $+1$.

Answer note for evaluators: a): It is custom to define a long-run multiplier as the change in the dependent variable with respect to permanent unit-change in a regressor (explanatory variable). Mathematically, the derivative of the steady-state equation of Y with respect to X.
b):

$$\beta_* = \frac{0.8 - 0.3}{1 - 0.5} = 1 \tag{2}$$

Mathematically, a long-run multiplier is equivalent to the infinite sum of the corresponding dynamic multipliers (sometimes called "lag-weights").

5. Explain why the validity of the above calculations of multipliers depends on X_t being a strongly exogenous variable.
Answer note for evaluators: Part of the definition of strong exogeneity is that there is no feed-back from Y_{t-1} to X_t., (one-way Granger causality) If this assumption does not hold, valid derivation of multipliers is based on a two-equation model where the two-way (joint) Granger causality is represented (put on model equation form). DEEEM Chapter 8. Lecture 8.

6. Assume that Y_t and X_t are jointly generated by a VAR with first order dynamics and with error-terms distributed as:

$$\begin{pmatrix} \varepsilon_{yt} \\ \varepsilon_{xt} \end{pmatrix} \sim IIN(0, \mathbf{\Sigma}),$$

where the matrix $\mathbf{\Sigma}$ has the two variances and the covariance of the error-terms as elements.

   (a) Why is (1) the conditional model equation of Y_t given X_t?
   (b) Are the OLS estimators of the coefficients of (1) consistent estimators? Explain your answer.

Answer note for evaluators: a) Y_t and X_t have a joint Normal distribution (conditional on Y_{t-1} and X_{t-1}). Therefore, the conditional expectation of Y_t given X_t is:

$$E(Y_t \mid X_t, X_{t-1}, X_{t-1})) = \phi_0 + \phi_1 Y_{t-1} + \beta_0 X_t + \beta_1 X_{t-1}$$

where the coefficients can be expressed in terms of the parameters of the VAR (in particular $\beta_0 = \sigma_{xy}/\sigma_x^2$), but the details about that is not required in this answer Defining the error term ϵ_t as:

$$\epsilon_t = Y_t - E(Y \mid X_t, Y_{t-1}, X_{t-1})$$

and therefore we get (1), which is the conditional model equation of Y_t given X_t.
If the $\mathbf{\Sigma}$ is diagonal, then $\beta_0 = 0$ is implied. b) Yes. Consistency of the OLS estimators of the coefficients depends on X_t, X_{t-1} and Y_{t-1} being pre-determined variables, meaning that they are uncorrelated with current and future error-terms: $\epsilon_t, \epsilon_{t+1}, \ldots$. Pre-determinedness follows from the assumptions about the two series being generated by the gaussian-VAR and by the conditioning on X_t which makes ϵ_t uncorrelated with X_t.

Question B (60 %)

1. In Table 1 you find results for ADF tests for three variables that are relevant for modelling wage-price dynamics of the US economy:

   LW: The natural logarithm of compensation per hour worked (in the non-agricultural business sector of the economy)

   DLW: The one-period change in LW (also called the first-difference of LW), ie., DLW = LW - LW_1, where LW_1 denotes the lagged LW.

   1/U: The reciprocal of the unemployment rate in the U.S. economy.

   The time series are quarterly.
   Explain why, by using the information in the table, it is reasonable to base modelling on the assumptions that *LW* is an I(1) variable while 1/U is an I(0) variable.
   Answer note for evaluators: The first set of results (for LW) shows that none of the ADF-tests are statistically significant, using the critical values in the table. (Based on the t-DY lags the D-lag 3 ADF is formally the most correct to report though). The middle set of result contains several significant ADF-test, and again the ADF in the row D-lag 3 is the most correct one, as t-DY_lag 4 is insignificant in the D-lag 4 row. Hence, it is reasonable to conclude that $DLW \sim I(1)$ *can be rejected. In the third section of the table there is support for rejection of* $1/U \sim I(1)$*, as the ADF in the D-lag 0 row is invalid in the light of the other tests.*

2. In the following we make use of three additional variables:

   PCE: Deflator of private consumption expenditure (consumer price index).

   Q: Deflator of value added in the non-agricultural business sector of the economy (producer price index).

   Value added (in real terms) per hour worked in the non-agricultural business sector of the economy (productivity).

   You can take as given that the natural logarithms of these variables, LPCE, LW and LZ, are I(1) variables.

   There is a long tradition for estimation of Wage Phillips Curve Models (W-PCM) on US time series data. Table 2 shows results for a W-PCM. They are reported with a single mis-specification test. This is to save space, you can take as given that none of the omitted standard mis-specification test indicate statistical residual mis-specification.

   (a) Interpret the AR 1-5 test and explain its importance for the consistency of the OLS estimators of the coefficients, and for the validity of using the standard errors in the judgement of the coefficients statistical significance.
   Answer note for evaluators: AR 1-5 tests the joint null hypothesis of no (autoregressive) residual autocorrelation between lag 1 and lag 5. Importance for consistency: If the test does not reject, the pre-determinedness of the lagged dependent variable is supported (which is necessary for consistency of OLS). If it rejects, the opposite follows: A lagged dependent variable is then correlated with future error-terms, not only lagged error-terms. Validity of using OLS standard error: They are based on the assumption of no residual autocorrelation. So if the AR-test rejects, the standard errors of coefficients can either be under-stated (leading to over-rejection, the case of positive residual autocorrelation) or over-stated (the case of negative autocorrelation). Usual "fix" is to use robust standard error. But note that this does no solve the problem of lagged Y's not being pre-determined when the AR-test is statistically significant.

(b) Explain why the results in Table 2 support a downward sloping wage Phillips curve.
Answer note for evaluators: The significance of $1/U$, implying the derivative $\frac{\partial DLW}{\partial U} = -0.026(1/U_1)^2 < 0$.

(c) The restricted model in Table 3 imposes two restrictions on the model in Table 2. Use information found in the two tables and calculate the following test statistic:

$$F(2, 204) = 0.4,$$

when one decimal point is used. The p-value is 0.6 (which you do not need to show). What is your conclusion about the statistical validity of the restrictions implied by the model in Table 3?
Answer note for evaluators:

$$\begin{aligned} F(2, 204) &= \frac{0.00753285130 - 0.00750126541}{0.00750126541} \times \frac{(213 - 9)}{2} \\ &= 0.4295 \end{aligned}$$

(d) What is the value of the Likelihood-Ratio (LR) test statistic in this case? Which distribution would you use to judge the statistical significance of the LR-test?
Answer note for evaluators:

$$LR = -2(789.367 - 789.815) = 0.896$$

Chi-square with 2 degrees of freedom.

3. Use the results in Table 4, together with the critical values of the ECM-test of cointegration found in Table 7, to test the null hypothesis that LW is not cointegrated with LZ and LQ. at a conventional level of significance.
Answer note for evaluators: The t-value of LW_1 can be compared to the critical values in Table 7, the case of $k = 3$, and the H_0 is therefore rejected at the 5% level.

4. Conditional on cointegration:

(a) What is the estimated long-run relationship for the wage variable LW?
Answer note for evaluators: ECM long-run relationship:

$$\begin{aligned} LW &= Const + \frac{0.0630023}{0.0612558} LQ + \frac{0.0464775}{0.0612558} LZ + \frac{0.0310781}{0.0612558} U^{-1} \\ &= Const + 1.03LQ + 0.76LZ + 0.51U^{-1} \end{aligned}$$

(b) Can you give an economic interpretation of the coefficients of the estimated equation?
Answer note for evaluators: The cointegration coefficients are long-run elasticities. The coefficient of LQ is close to one and implies that a one percent increase in Q "lead to" a one percent increase in W. The coefficient of LZ is numerically smaller than one, so the trend like increase in Z implies a secular decline in the wage-share: (LW-Q-LZ)=-0.24LZ)

5. The ECM-test of cointegration used above assumes that LQ and LZ are weakly exogenous with respect to the cointegration coefficients. Can you explain in words the meaning of this assumption?
Answer note for evaluators: Only the wage variable should equilibrium correct to deviations from the long-run relationship. Implicitly there are zero-restrictions on the adjustment coefficients of DLQ and DLZ.

6. Table 5 contains results for the Johansen-method to cointegration analysis. The results were based on the estimation of a VAR with four endogenous variables: LW, LZ, LQ and (1/U). The VAR had second order dynamics, and it was not mis-specified. Explain why the output support that the number of cointegration vectors (r) can be set to $r = 2$.
   Answer note for evaluators: The Trace test rejects the H_0 of $r = 0$ against H_1 of $r \leq 1$. Conditional on that also H_0 of $r = 1$ against H_1 of $r \leq 2$ is rejected. But not H_0: $r = 2$ against H_1: $r \leq 3$

7. Table 6 shows the two vectors with cointegration coefficients (`beta` in the results).

   (a) Explain why the two vectors (and hence the two long-run relationships between the variables) are not identified without making further assumptions.
   Answer note for evaluators: What we have here is a particular special case of multiple cointegration. From cointegration theory (and the symbols we have used in the book and in the lectures) we know that for the factorization $\Pi = \alpha\beta'$, β is not uniquely determined from a known Π (even when each of the vectors are normalized by setting one element in each to 1), without making further assumptions, which can be about α, as well as about β, or both. In the lectures and in the book, the point was made that the lack of generic identification is analogous to the identification problem of simultaneous equations that was studied earlier in the course. The point can be made by writing the two long-run relationships as a pair of simultaneous equations, and explain why neither of the two long-run relationships are identified on the order condition. For example: Write

$$LW + \beta_{11}LZ + \beta_{12}LQ + \beta_{13}1/U = e_{1t}$$
$$\beta_{21}LW + 1LZ + \beta_{22}LQ + \beta_{23}1/U = e_{2t}$$

   where e_{1t} and e_{2t} are I(0), then make the point that neither of the equations are identified in the order condition. Further assumptions must be made, for example in the form of linear restrictions on the βs. One example, which is consistent with the evidence earlier in QB could be $\beta_{21} = \beta_{22} = 0$, since $(1/U)$ is I(0) earlier in QB, and $\beta_{12} = -1$ which would imply homogeneity of degree 1 in a wage-equation interpretation of the vector. These specific examples are not required to get a full score, though.

   (b) Given that the first vector can be identified, comment on the differences and similarities between this long-run relationship, and the long-run relationship that was estimated above (in QB4) by using the ECM-method to cointegration.
   Answer note for evaluators: Different estimators. In the Johansen method weak exogeneity is not assumed.

Tables with estimation results and facsimile of table with critical values for ECM-test

Table 1: Dickey-Fuller tests of unit-root. LW_t, DLW_t and $1/U_t$.

```
Unit-root tests
The sample is: 1967(1) - 2020(1)

LW: ADF tests (T=213, Constant+Trend; 5%=-3.43 1%=-4.00)
D-lag    t-adf      t-DY_lag  t-prob
 4     -1.634        1.511    0.1321
 3     -1.586        2.251    0.0255
 2     -1.559        3.138    0.0019
 1     -1.573        0.1215   0.9034
 0     -1.581

DLW: ADF tests (T=213, Constant; 5%=-2.88 1%=-3.46)
D-lag    t-adf      t-DY_lag  t-prob
 4     -2.755       -0.9240   0.3562
 3     -3.327*      -3.111    0.0021
 2     -4.302**     -3.616    0.0004
 1     -5.971**     -5.698    0.0000
 0     -11.17**


 1/U: ADF tests (T=213, Constant; 5%=-2.88 1%=-3.46)
D-lag    t-adf      t-DY_lag  t-prob
 4     -3.052*     -0.6179    0.5373
 3     -3.695**     0.9569    0.3397
 2     -3.571**     3.514     0.0005
 1     -2.889*      10.76     0.0000
 0     -1.435
```

Table 2: Estimation results for a W-PCM.

Modelling DLW by OLS
The estimation sample is: 1967(1) - 2020(1)

	Coefficient	Std.Error	t-value	t-prob
DLW_1	-0.0608953	0.06760	-0.901	0.3688
Constant	-0.000363902	0.002044	-0.178	0.8589
DLPCE	0.315005	0.1577	2.00	0.0471
DLPCE_1	0.0429881	0.1954	0.220	0.8261
DLZ	0.222952	0.06181	3.61	0.0004
DLZ_1	0.0927401	0.06241	1.49	0.1389
DLQ	0.188824	0.1706	1.11	0.2698
DLQ_1	0.342539	0.1548	2.21	0.0281
1/U_1	0.0259039	0.009361	2.77	0.0062

sigma	0.00617070	RSS	0.00750126541
Adj.R^2	0.516377	log-likelihood	789.815
no. of observations	213	no. of parameters	9

AR 1-5 test: F(5,192) = 0.76525 [0.5759]

Table 3: Estimation results for a simplified W-PCM.

Modelling DLW by OLS
The estimation sample is: 1967(1) - 2020(1)

		Coefficient	Std.Error	t-value	t-prob
Constant		-0.000150620	0.001981	-0.0760	0.9395
DLPCE		0.298481	0.1485	2.01	0.0458
DLZ		0.209849	0.05994	3.50	0.0006
DLZ_1		0.0783854	0.06018	1.30	0.1943
DLQ		0.188300	0.1680	1.12	0.2636
DLQ_1		0.343052	0.1110	3.09	0.0023
1/U_1	U	0.0242042	0.009116	2.66	0.0086

sigma	0.00615252	RSS	0.00753285130
Adj.R^2	0.519222	log-likelihood	789.367
no. of observations	213	no. of parameters	7

AR 1-5 test: F(5,194) = 0.42876 [0.8282]

Table 4: Estimation results for a ECM equation for wages.

Modelling DLW by OLS
The estimation sample is: 1967(1) - 2020(1)

	Coefficient	Std.Error	t-value	t-prob
DLW_1	-0.0668265	0.06720	-0.994	0.3213
Constant	-0.00195481	0.002613	-0.748	0.4553
DLPCE	0.253588	0.1562	1.62	0.1062
DLPCE_1	0.0887093	0.1945	0.456	0.6489
DLZ	0.239360	0.06373	3.76	0.0002
DLZ_1	0.0719792	0.06171	1.17	0.2449
DLQ	0.194162	0.1683	1.15	0.2501
DLQ_1	0.230589	0.1539	1.50	0.1357
LW_1	-0.0612558	0.01726	-3.55	0.0005
LQ_1	0.0630023	0.02756	2.29	0.0233
LZ_1	0.0464775	0.02060	2.26	0.0252
1/U_1	0.0310781	0.01096	2.83	0.0051

sigma	0.00600156	RSS	0.00698764248
Adj.R^2	0.542525	log-likelihood	797.368
no. of observations	213	no. of parameters	12

AR 1-5 test: F(5,189) = 0.81534 [0.5401]

Table 5: Johansen method: Cointegration rank test results.

I(1) cointegration analysis, 1967(1) - 2020(1)

eigenvalue	loglik for	rank
	3155.239	0
0.14025	3171.332	1
0.12231	3185.226	2
0.073521	3193.359	3
0.0057715	3193.975	4

H0:rank<=	Trace test	[Prob]	
0	77.472	[0.000]	**
1	45.286	[0.000]	**
2	15.498	[0.053]	
3	1.2329	[0.267]	

Asymptotic p-values based on: Unrestricted constant

Table 6: Johansen method: Estimated $\boldsymbol{\beta}$-vectors.

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Cointegrated VAR
       The estimation sample is: 1967(1) - 2020(1)

Cointegrated VAR (2) in:
[0] = LW
[1] = LZ
[2] = LQ
[3] = 1/U
Unrestricted variables:
[0] = Constant
Number of lags used in the analysis: 2

beta
LW            1.0000      -1.4795
LZ          -0.79029       1.0000
LQ           -1.0977       1.6211
1/U         -0.16286       1.2347

alpha
LW          0.046915     0.039879
LZ          0.060176   -0.0065990
LQ          0.047703   -0.0035823
1/U        -0.049645    -0.012984
```

Table 7: Facsimile from article by Ericsson and MacKinnon.

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Table 3. Response surface estimates for critical values of the ECM test of cointegration $\kappa_c(k)$: with a constant term.

k	Size (%)	θ_∞	(s.e.)	θ_1	θ_2	θ_3	$\hat{\sigma}$
1	1	−3.4307	(0.0006)	−6.52	−4.7	−10	0.00790
	5	−2.8617	(0.0003)	−2.81	−3.2	37	0.00431
	10	−2.5668	(0.0003)	−1.56	2.1	−29	0.00332
2	1	−3.7948	(0.0006)	−7.87	−3.6	−28	0.00847
	5	−3.2145	(0.0003)	−3.21	−2.0	17	0.00438
	10	−2.9083	(0.0002)	−1.55	1.9	−25	0.00338
3	1	−4.0947	(0.0005)	−8.59	−2.0	−65	0.00857
	5	−3.5057	(0.0003)	−3.27	1.1	−34	0.00462
	10	−3.1924	(0.0002)	−1.23	2.1	−39	0.00364
[illegible]	1	−4.3555	(0.0006)	−8.90	−6.7	−31	[illegible]