Solution to exercises, 4230/4235

7.4.

The indirect utility function:

$$v(p_1, p_2, m) = \frac{m}{p_1 + p_2}$$

(a) The demand functions

Interpretation: the Marshall demand functions $x_i(p_1,p_2,m)$

Use Roy's identity:

$$x_i(p_1, p_2, m) = -\frac{\frac{\partial v(p_1, p_2, m)}{\partial p_i}}{\frac{\partial v(p_1, p_2, m)}{\partial m}} = \frac{\frac{m}{(p_1 + p_2)^2}}{\frac{1}{p_1 + p_2}} = \frac{m}{p_1 + p_2},\ i = 1, 2$$

NB! The demand functions are equal for the two goods, i.e. the optimal choice is identical quantities of the two goods, independent of relative prices. The optimal choices is on the 45o line in the goods space, and the indifference curves have their corner on this line and extend vertically and horizontally from the points on the line.

(b) The expenditure function

The indirect utility function is found as a solution to the following problem:

$$v(p_1, p_2, m) = Max\, u(x_1, x_2)\, s.t. \sum_{i=1}^{2} p_i x_i \leq m$$

The expenditure function is found as a solution to the following problem:

$$e(p_1, p_2, u) = Min \sum_{i=1}^{2} p_i x_i\ s.t.\ u(x_1, x_2) \geq u^o$$

If we choose as the given utility level in the expenditure problem the solution to the utility maximization problem for given income m we have in general :

$$v(p_1, p_2, m) = v(p_1, p_2, e(p_1, p_2, u)) = u$$

Using the given indirect utility function above yields:

$$v(p_1, p_2, m) = \frac{m}{p_1 + p_2} = u \Rightarrow m = u(p_1 + p_2) \Rightarrow$$

$$e(p_1, p_2, u) = u(p_1 + p_2)$$

(c) The direct utility function:

The shape of the indifference curves tells us that if we have more of one good than the other the utility level is determined by the smallest amount (it does not increase our utility to get more than one right hand shoe if we only have one left hand shoe). Therefore the utility function is of the form:

$$u(x_1, x_2) = Min\{x_1, x_2\}$$

8.2.
Interpretation of the C-D demand system:
The utility function is C-D with constant returns to scale:
$u(x_1, x_2) = x_1{}^a x_2{}^{1-a}, a \in (0,1)$
To find the substitution effects we have to know that the change in demand (Marshall demand function) when a price change can be decomposed into a substitution effect; change in Hicksian (compensated) demand function, and an income effect; the Slutsky equation.. We therefore have to derive the Hicks demand functions. These function are found by solving the minimum expenditure problem for a given utility level;

$$e(p_1, p_2, u) = Min \sum_{i=1}^{2} p_i x_i \ s.t.\ u(x_1, x_2) \geq u^o$$

The Lagrangian for the problem is:

$$L = \sum_{i=1}^{2} p_i x_i - \theta(x_1{}^a x_2{}^{1-a} - u^o)$$

Differentiating yields:

$$\frac{\partial L}{\partial x_1} = p_1 - \theta a x_1{}^{a-1} x_2{}^{1-a} = 0,$$

$$\frac{\partial L}{\partial x_1} = p_2 - \theta(1-a) x_1{}^a x_2{}^{2-a} = 0$$

$$\Rightarrow p_1 = \theta a \frac{u^o}{x_1}, p_2 = \theta(1-a)\frac{u^o}{x_2}$$

Eliminating the Lagrangian parameter θ by dividing the two equations on each side yields:

$$\frac{p_1}{p_2} = \frac{a\frac{u^o}{x_1}}{(1-a)\frac{u^o}{x_2}} = \frac{a}{1-a}\frac{x_2}{x_1}$$

Solving for x_1 yields:

$$x_1 = \frac{a}{1-a}\frac{p_2}{p_1} x_2$$

The second equation between x_1 and x_2 is the constant level of the utility function. Solving for x_2 yields:

$$x_1{}^a x_2{}^{1-a} = u^o \Rightarrow x_2{}^{1-a} = u^o x_1{}^{-a} \Rightarrow$$

$$x_2 = (u^o x_1{}^{-a})^{\frac{1}{1-a}} = u^{o\frac{1}{1-a}} x_1{}^{\frac{-a}{1-a}}$$

Substituting for x_2 above yields:

$$x_1 = \frac{a}{1-a}\frac{p_2}{p_1}x_2 = \frac{a}{1-a}\frac{p_2}{p_1}u^{o\frac{1}{1-a}}x_1^{\frac{-a}{1-a}} \Rightarrow$$

$$x_1^{\frac{1}{1-a}} = \frac{a}{1-a}\frac{p_2}{p_1}u^{o\frac{1}{1-a}} \Rightarrow$$

$$x_1 = \left[\frac{a}{1-a}\frac{p_2}{p_1}u^{o\frac{1}{1-a}}\right]^{1-a} = \left[\frac{a}{1-a}\right]^{1-a}\left[\frac{p_2}{p_1}\right]^{1-a} \cdot u^o = h_1(p_1, p_2, u)$$

The Hicks demand function for x_2 is found by substitution:

$$x_2 = (u^o x_1^{-a})^{\frac{1}{1-a}} = u^{o\frac{1}{1-a}}x_1^{\frac{-a}{1-a}} = u^{o\frac{1}{1-a}}\left[\left[\frac{a}{1-a}\right]^{1-a}\left[\frac{p_2}{p_1}\right]^{1-a} \cdot u^o\right]^{\frac{-a}{1-a}} = \left[\frac{a}{1-a}\right]^{-a}\left[\frac{p_2}{p_1}\right]^{-a} \cdot u^o$$

$$= h_2(p_1, p_2, u)$$

The substitution matrix:

$$\begin{bmatrix} \frac{\partial h_1}{\partial p_1} \frac{\partial h_1}{\partial p_2} \\ \frac{\partial h_2}{\partial p_1} \frac{\partial h_2}{\partial p_2} \end{bmatrix}$$

$$\frac{\partial h_1}{\partial p_1} = \left[\frac{a}{1-a}\right]^{1-a} \cdot u \cdot (-)(1-a)p_2^{1-a}p_1^{-(2-a)} < 0$$

$$\frac{\partial h_1}{\partial p_2} = \left[\frac{a}{1-a}\right]^{1-a} \cdot u(1-a)p_2^{-a}p_1^{-(1-a)} = a^{1-a}(1-a)^a \cdot u \cdot p_2^{-a}p_1^{-(1-a)} > 0$$

$$\frac{\partial h_2}{\partial p_1} = \left[\frac{a}{1-a}\right]^{-a} \cdot u \cdot ap_2^{-a}p_1^{-(1-a)} = a^{1-a}(1-a)^a \cdot u \cdot p_2^{-a}p_1^{-(1-a)} > 0$$

$$\frac{\partial h_2}{\partial p_2} = \left[\frac{a}{1-a}\right]^{-a} \cdot u(-a)p_2^{-(1+a)}p_1^{-a} < 0$$

The diagonal terms are negative and cross-price effects are symmetric; $\frac{\partial h_1}{\partial p_2} = \frac{\partial h_2}{\partial p_1}$.