4240 - Equilibrium, Welfare & Information
Make Up Exam, Fall Term 2019

Problem 1

i) Define a Walrasian equlibrium and state the second welfare theorem of economics.

> **Solution:** A Walrasian equilibrium is an allocation of resources and an associated price vector p^* such that quantity demanded equals the quantity supplied.
>
> The second theorem of welfare economics states that for any Pareto efficient allocation of resources, there exists a set of initial endowments and a related price vector such that this allocation is a Walrasian equilibrium.

Problem 2 [each of i-v carries equal weight]
Two agents, denoted a and b, live in a 2-commodity exchange economy and have the following utility functions:

$$\begin{aligned} U(x_a, y_a) &= x_a y_a \\ U(x_b, y_b) &= \alpha x_b - \beta x_b^2 + y_b \end{aligned}$$

Initial endowments are $w_a = (1, \frac{1}{2})$ and $w_b = (1, \frac{3}{2})$. All agents are price takers. Please normalize the price for good y to unity.

i) Assume $\alpha, \beta > 0$. Derive the Walrasian aggregate demand for good x as a function of price p_x and individual wealth levels $m_i, i \in \{a, b\}$.

Solution:

Let $m_i = p_x \bar{x}_i + \bar{y}_i$ denote the budget of person $\in \{a, b\}$ ($p_y = 1$). Substituting the budget constraint in to eliminate y_a, individual a maximizes

$$U(x_a, y_a) = x_a (m_a - p_x x_a)$$

w.r.t. x_a and similarly individual b maximizes

$$U(x_b, y_b) = \alpha x_b - \beta x_b^2 + m_b - p_x x_b$$

w.r.t. x_b. We find the demand functions

$$x_a(p_x, m_a) = \frac{m_a}{2p_x} \quad \text{and} \quad x_b(p_x) = \frac{\alpha - p_x}{2\beta}$$

giving us the aggregate demand

$$x(p_x, m_a) = \frac{m_a}{2p_x} + \frac{\alpha - p_x}{2\beta}$$

which is independent of m_b.

ii) What is the aggregate demand for good x as a function of only the price p_x?

Solution: Using the initial endowments we find $m_a = 1p_x + \frac{1}{2}$ and

$$\begin{aligned} x(p_x) &= \frac{1}{2} + \frac{1}{4p_x} + \frac{\alpha - p_x}{2\beta} \\ &= \frac{2\beta p_x + \beta + 2\alpha p_x - 2p_x^2}{4\beta p_x} \end{aligned}$$

iii) Assume that $\alpha = 6$ and $\beta = 2$. What are the equilibrium allocation of good x and the equilibrium price?

Solution: Equating supply and demand delivers

$$\begin{aligned} & 2 = \tfrac{2\beta p_x + \beta + 2\alpha p_x - 2p_x^2}{4\beta p_x} \\ \Leftrightarrow \quad & 8\beta p_x = 2\beta p_x + \beta + 2\alpha p_x - 2p_x^2 \\ \Leftrightarrow \quad & p_x^2 + (4\beta - \beta - \alpha)p_x - \tfrac{1}{2}\beta = 0 \end{aligned}$$

and using $\alpha = 6$ and $\beta = 2$ we have

$$\begin{aligned} & p_x^2 + (8 - 2 - 6)p_x - 1 = 0 \\ \Rightarrow \quad & p_x^* = 1 \end{aligned}$$

The equilibrium allocations of good x are accordingly

$$\begin{aligned} x_a^* &= \frac{p_x + \frac{1}{2}}{2p_x} = \frac{3}{4} \\ x_b^* &= \frac{6-1}{4} = \frac{5}{4} = 2 - x_a^*. \end{aligned}$$

Note that the equilibrium price for the second good is $p_y^* = 1$ by normalization.

iv) Assume that $\alpha = 2$ and $\beta = 0$. What is individual a's Walrasian demand for good x?

Solution:

NOTE: The question was meant to be about agent b and to read: "What is individual b's Walrasian demand for good x?". Given the typo, the correct statement of either demand function was accepted.[1]

Demand of agent a: Indvidual a's demand is

$$x_a(p_x, m_a) = \frac{m_a}{2p_x} = \frac{p_x + \frac{1}{2}}{2p_x}$$

where I inserted the individual's endowment to obtain his or her wealth.

Demand of agent b: Then the utility function of individual b is linear:

$$U(x_b, y_b) = 2x_b + y_b = 2x_b + m_b - p_x x_b$$

The agent is indifferent between consuming x and y if and only if the price is $p_x = 2$, which can be seen e.g. from the derivative of the above utility

$$2 - p_x = 0 \Leftrightarrow p_x = 2.$$

However, the individual cannot control the price. If $p_x > 2$ the individual would consume only y, and if $p_x < 2$ the individual would consume only x. Hence, the demand function of individual a is

$$x_b(p_x, m_b) \begin{cases} = \dfrac{m_b}{p_x} & \text{if } p_x < 2 \\ \in \left[0, \dfrac{m_b}{2}\right] & \text{if } p_x = 2 \\ = 0 & \text{if } p_x > 2 \end{cases}$$

Verbal descriptions of the case $p_x = 2$ are fine. Relating it to expenditure on good y is of course also welcome.

v) Assume that $\alpha = 2$ and $\beta = 0$. What are the equilibrium allocation of good x and the equilibrium price?

Solution: Indvidual a's demand is

$$x_a(p_x, m_a) = \frac{m_a}{2p_x} = \frac{p_x + \frac{1}{2}}{2p_x}$$

where I inserted the individual's endowment to obtain his or her wealth. Doing the same for individual b's demand function yields

$$x_b(p_x) \begin{cases} = \dfrac{p_x + \frac{3}{2}}{p_x} & \text{if } p_x < 2 \\ \in \left[0, \dfrac{p_x + \frac{3}{2}}{2}\right] & \text{if } p_x = 2 \\ = 0 & \text{if } p_x > 2 \end{cases}$$

The total supply of good x is $\bar{x} = 2$. For a price $p_x \neq 2$ aggregate demand is

$$x(p_x) = \begin{cases} \dfrac{p_x + \frac{1}{2} + 2p_x + 3}{2p_x} = \dfrac{3p_x + \frac{1}{2} + 3}{2p_x} & \text{if } p_x < 2 \\ \dfrac{p_x + \frac{1}{2}}{2p_x} & \text{if } p_x > 2 \end{cases}$$

which cannot meet the supply: the case $p_x < 2$ would only yield an equilibrium for $p_x = \frac{7}{2} > 2$, a contradiction, and the case $p_x < 2$ would only yield an equilibrium for $p_x = \frac{1}{6} < 2$, also a contradiction.

Thus, the equilibrium must be in the interior and $p_x^* = 2$. Then the equilbrium allocation is determined by individual a's demand as

$$x_a(p_x^*) = \frac{2 + \frac{1}{2}}{4} = \frac{5}{8}$$

leaving the amount

$$x_b^* = 2 - \frac{5}{8} = \frac{11}{8}$$

for individual a who is indifferent between spending his or her budget on good x or y.