→ Revised slides for April 13th & April 20th

→ The 1x 45min lecture on April 20th had a fair deal on
- reviewing & elaborating upon
- correcting :(

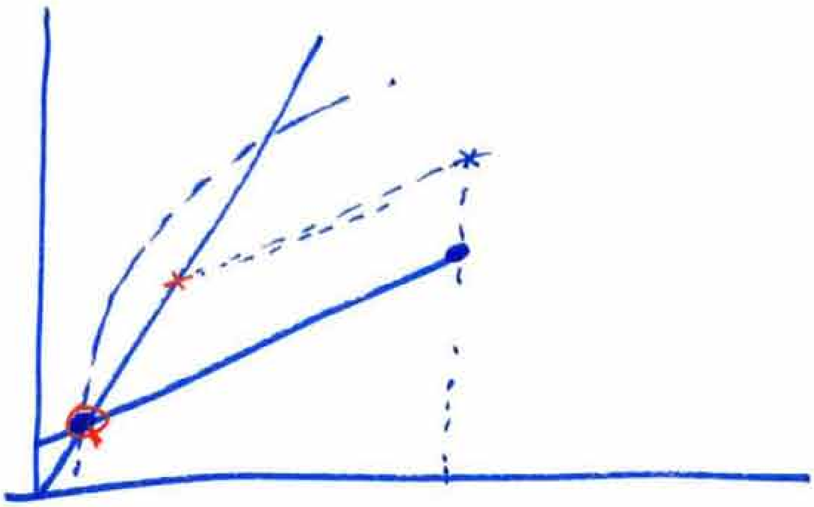
the April 13th lecture.

→ So I have merged in the April 20th slides into the April 13th slides, trying to put each new slide in where it seemed most fitting.

→ Some slides deleted, one new (bad scan), some headlines confusing – see annotations made (using red typewriter font like this).

Hope this is clearer, despite all.
Sorry for the inconvenience! – Nils

2.12 Renegotiation



We might argue that

- *if* type is revealed to P just after contract, then P may renegotiate with the inefficient agent to get ⋆ rather than ∘
- But: the efficient will have incentives to mimic. (or lie.)

Whether type can be hidden "all the way", is a signaling issue.

Moral hazard (ch 4)

→ * **How** to induce (unverifiable) effort?
→ * **Whether** to induce effort?

Compared to ch 2:

→ we now assume one type of agent
→ **stochastic** output
→ ... whose probabilities affected by the agent's **effort**
→ ... which is costly for the agent

Ch 2	Ch 4
→ P chooses contract; menu item (q, t) pair	Possible q-levels given. Contract: t(q).
→ Accepted (or not) and if accepted:	
→ Contract ⇝ efficient agent may choose to mimic ineff. ag.	Contract ⇝ choice of effort level
→ q follows (deterministic!)	q drawn, stochastically dependent on effort. (Agent may "hide" production and mimic lower level)
→ Contract fulfilled	

Basics on the contract

→ The possible output levels $\{q_i\}$ are given, but – regardless of actions – there is a positive probability of a "low" output level

→ (P) chooses $t(q)$.
I.e.: t_i for q_i, all possible q_i.

→ Notation:
- → The book uses $\bar{q}$ and $\underline{q}$ with $\bar{q} > \underline{q}$ – i.e. same notation as ch 2 but opposite order than under S – M.
- → From 4.5, book uses q_i, increasing – same notation as I used, but opposite order
- → I choose: $q_1 > q_0$ as 4.5;
  (later $q_n > q_{n-1} > \ldots > q_1 > q_0$)
- → $\Delta q = q_1 - q_0$ is > 0.

Effort:

→ ~~[illegible]~~ Two levels, $e \in \{0, 1\}$

→ $e = 1$ increases probability of the better output q_1.

Probabilities: (stoch.)

$$\pi_i = Pr[q = q_1 \mid e = i]$$

assumed so that

$$\Delta\pi := \pi_1 - \pi_0 \quad \text{is} \quad > 0.$$

→ Effort is "bad" for Ⓐ.

Disutility:

$$\left.\begin{array}{ll} \psi & \text{if } e = 1 \\ 0 & \text{if } e = 0 \end{array}\right\} \Rightarrow \psi \cdot e$$

Preferences

Principal:

By first-order stoch. dominance, any (P) who prefers more to less, will prefer that effort is exercised (as long as P does not have to pay to induce it!)

Nevertheless: <u>will only treat risk-neutral principal!</u>

Return: s_i for q_i.

Agent:

Utility = $u(t) - \psi \cdot e$,

with $u' > 0 \geq u''$

The book assumes $u(0) = 0$, i.e. that the only "cost" is due to effort. Should not be hard to generalize.

u strictly increasing, so has an inverse. Notation:

$h = u^{-1}$

Benchmark case: verifiable effort.

1) $e = 1$ contractual:

Part. constraint

$$\underbrace{\pi_1 u(t_1) + (1-\pi_1) u(t_0)}_{\text{shorthand notation}} - \psi \geq 0$$

$$E_1 [u(t)]$$

"1": expectation using the probabilities which follow from $e=1$.
("E_0": analogously)

Principal solves

$$\max_{t_0, t_1} \pi_1 (s_1 - t_1) + (1-\pi_1)(s_0 - t_0)$$

s.t. participation.

K-T 1st order cond's $\rightsquigarrow$ $\underline{t_0 = t_1} = t^*$

(P fully insures A!)

Also: participation binding

so $u(t^*) = \psi$

$\underline{t^* = h(\psi)}$

2) $e=0$ contractual:[(*)]

$$\max_{t_0, t_1} E_0[s-t] \quad \text{s.t} \quad E_0[u(t)]=0$$

also gives full insurance: $t_0 = t_1 = h(0)$
which is zero

So Ⓟ will ~~not offer~~ induce effort if

$$E_1[s] - h(\psi) > E_0[s]$$

i.e.

$$h(\psi) < \Delta\pi \cdot \Delta s$$

where $\Delta s = s_1 - s_0 > 0$.

$B := \Delta\pi \; \Delta s$ is the benefit of effort.

(*): If Ⓟ and Ⓐ have a <u>common interest</u> in effort, then we have no problem to study.

Effort not verifiable – must be induced if desirable.

The moral hazard incentive constraint:

$$E_1[u(t)] - \psi \geq E_0[u(t)]$$

Then incentive feasibility reads:

$$E_1[u(t)] - \psi \geq \begin{cases} E_0[u(t)] \\ 0 \end{cases}$$

We now formulate the problem <u>assuming</u> that Ⓟ wants to induce effort; has to be compared to case where effort is <u>not</u> induced

corr Apr 20

Assuming it is optimal to induce effort:

Principal solves

$$E_1[s-t] \quad \text{s.t. inc. feas.}$$

Shall treat two cases: (A) risk neutral and (B) strictly risk averse.

Will turn out: In both cases, both inc. feas. constraints ~~are binding~~ "can be chosen with equality" -- Apr 20

This leads to

$$\begin{bmatrix} 1-\pi_0 & \pi_0 \\ 1-\pi_1 & \pi_1 \end{bmatrix} \begin{bmatrix} u(t_0) \\ u(t_1) \end{bmatrix} = \begin{bmatrix} 0 \\ \psi \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u(t_0) \\ u(t_1) \end{bmatrix} = \begin{pmatrix} 1 - \frac{\pi_1}{\Delta\pi} \\ 1 + \frac{1-\pi_1}{\Delta\pi} \end{pmatrix} \psi$$

(compare to full information case with ψ for both.)

Note: $u(t_0) = -\frac{\pi_0}{\Delta\pi}\psi < 0$.

corr Apr 20

(when) is it optimal to induce effort?

Benefit: still

$$B = E_1[s] - E_0[s] = \Delta\pi \, \Delta s$$

Cost

$$E_1[t] = E_1[h(u)]$$

$$\overset{(*)}{\geq} h(E_1[u]) = \underbrace{h(\psi)}_{\text{= cost in full information case}}$$

"binding" constr. (should be "active", i.e. "with equality") -- Apr 20

(*) by Jensen's ineq:

= if risk neutral

> if strictly risk averse.

So: * with risk neutral agent, the full information first-best effort is implemented.

* with strictly risk-averse agent, there will be a non-empty interval $(h(\psi), E_1[t])$ where $e=0$ will be implemented but is socially inefficient.

Corr Apr 20

Proof that both constraints ~~are~~ can be chosen ~~binding~~ active:

(will both be binding in the strictly risk averse case though.)

→ Risk neutral case: can put $u(t) = t$.

Linear programming problem, nonzero determinant, corner solution.

See next slide (added Apr. 20)

→ Strictly risk averse case:

The transformation $t = h(u)$ leads to a concave program. K-T both necessary(*) and sufficient.

Solving 1st o.c. for the multipliers λ, μ yields

$$\lambda \cdot \underbrace{\Delta\pi}_{+} = \underbrace{\pi_1}_{+}\underbrace{(1-\pi_1)}_{+}\underbrace{\left(h'(u_1) - h'(u_0)\right)}_{\text{must be } \geq 0} > 0$$

Cannot be $= 0$, for then u = constant, $\Rightarrow \psi = 0$.

$$\mu = E_1\left[h'(u)\right] > 0 \text{ since } u' > 0.$$

(*) (Don't worry about the constraint qualification -- it holds.)

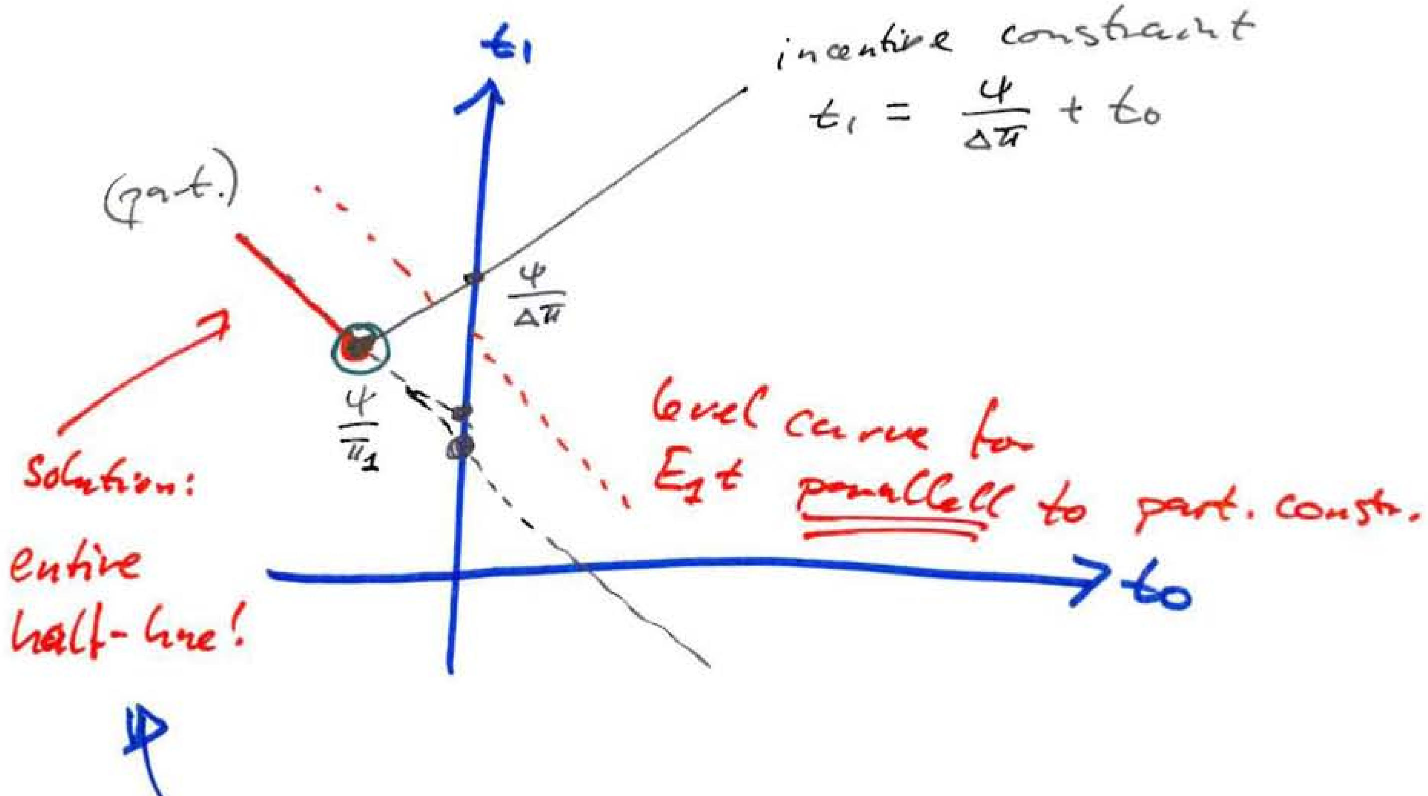
Linear programming -- graphical argument:

Two states, two variables:

$$\min E_1 t$$

$$\text{s.t. } E_1 t \geq \psi \qquad \text{(part. constr.)}$$

$$E_1 t - E_0 t \geq \psi \qquad \text{(inc. constr.)}$$



* Note: $\frac{\psi}{\Delta\pi} > \frac{\psi}{\pi_1}$ so "entire half-line" in 4th quadrant ($t_1 > 0 > t_0$)
* Point ∘ last to remain feasible under limited liability:

Limited liability constraints - 4.3

Assume: risk neutral agent

: an exogenous lower bound $-\ell$ on t.

* If $t_0 \geq -\ell$, then automatically $t_1 \geq -\ell$ too.

* Risk neutrality, $u(t) = t$

⇝ only interesting case is

if $-\frac{\pi_0}{\Delta\pi}\psi \leq -\ell$.

Then $t_0 \geq -\ell$ is binding;

Still, a linear program:

Solution at corner where

$t_0 = -\ell$

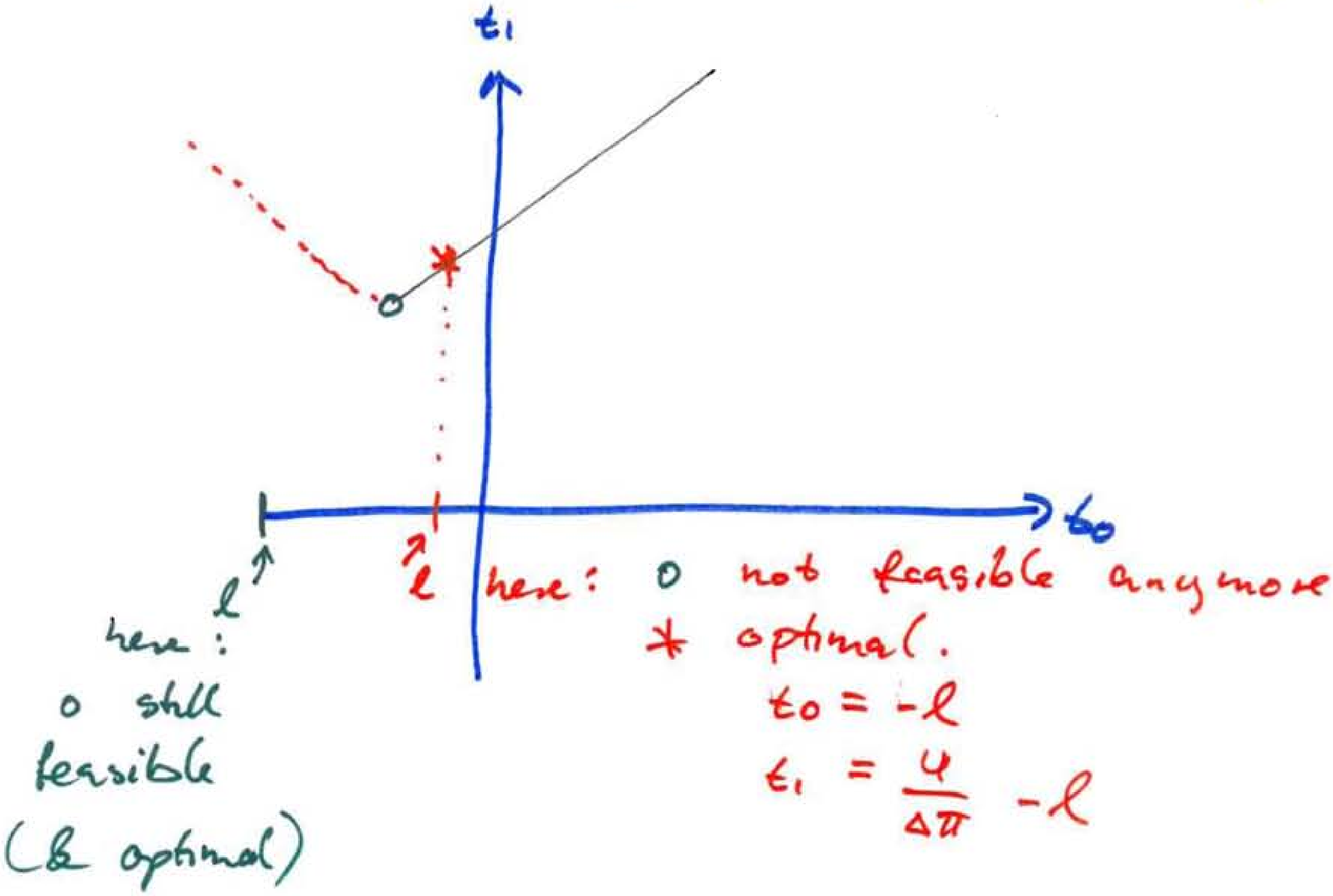
⋮

Linear programming:

Two states, two variables, limited liability

$t_0 \geq -\ell$
$t_1 \geq -\ell$ } same ℓ

(exogenously given constraints.)



$t_0 = -\ell$

$t_1 = \frac{4}{\Delta\pi} - \ell$

($t_1 > -\ell$ automatically.)

More than two outcomes — 4.5

* Keep the assumption that effort $\in \{0, 1\}$
* Assume the possible outcomes are

$$q_0 < q_1 < \ldots < q_{n-1} \qquad (n \text{ distinct})$$

(book : $q_1 < \ldots < q_n$)

with probabilities

$$\pi_{mi} = Pr\,[\,q = q_m \mid \text{effort} = i\,], \qquad i = 0, 1$$

$$\Delta\pi_m = \pi_{m1} - \pi_{m0}$$ ~~assumed > 0~~.

* Return to principal : S_m for q_m
* Contract : $t_m = t(q_m)$ $\qquad m = 0, \ldots, n-1$

The risk neutral case:

* The case without limited liability is again an LP problem with # eq's = # var's (counting multipliers).

* What if we are constrained to $t \geq 0$ (i.e: $t_m \geq 0$, all m) ?

Then participation <u>will not</u> be binding.

In order to induce effort:

$$\max E_1[s - t] \quad \text{s.t.} \begin{cases} E_1 t - E_0 t \geq \psi \\ t_m \geq 0 \end{cases}$$

multiplier

1^{st} o.c. (t_m):

$$-\pi_{m1} + \lambda \Delta\pi_m + \xi_m = 0,$$

Slackness: $\lambda \geq 0$, $\lambda = 0$ if $E_1 t - E_0 t > \psi$

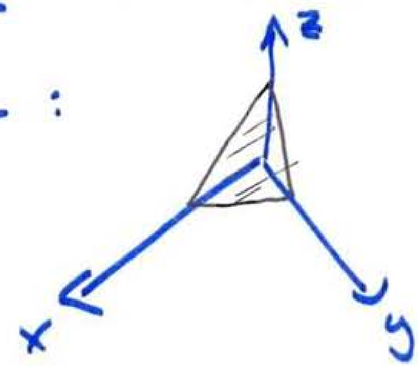
$$\xi_m \geq 0, \quad t_m \xi_m = 0$$

More states, more variables

$$E_1 t \geq 4$$

↑

all coeff's > 0 ; ~~[illegible]~~ cuts through 1^{st} orthant in this manner:



$$E_1 t - E_0 t \geq 4$$

$$= \sum \Delta\pi_m t_m$$

positive for some m, negative for some m

↳ neg. coeff: "might reduce as much as possible" (maybe together with one with pos. coeff) reduces payment.

* If no limited liability constraints:

  $E_1 t = \Psi$, not unique optimum.

* If limited liability constraints $t_i \geq 0$, all i ($\ell = 0$ for simplicity):

  - If solution, then at corner is possible
  - "Corners" are all on axes
  - On axis $\Rightarrow t_i = 0$ for all i except one.
  - Turns out: There is solution. Only one quantity rewarded: the one with highest $\frac{\Delta\pi_i}{\pi_i}$.
  - Troublesome (breaks model(?)) if this is not the highest q.

Analytical approach from March 13:

So:

$$\lambda \cdot \Delta\pi_m \leq \pi_{m1}$$

with $=$ if $t_m > 0$.

- We cannot have all $t_m = 0$ ($\Rightarrow$ no effort!)

- If $\Delta\pi_m \leq 0$ then $\lambda \Delta\pi_m \leq \pi_{m1}$ automatically.

But among those m for which $\Delta\pi_m > 0$, we must have

$$\lambda \leq \frac{\pi_{m1}}{\Delta\pi_m}$$ for them all, i.e. also for the smallest.

This leads to:

$$\lambda^{-1} = \max_{m=0,\dots,n-1} \frac{\Delta\pi_m}{\pi_{m1}}$$

and only the argmax m^* is (are) paid for! Assume m^* unique.

Solution:

P pays

$t_m = 0$ except

$t_{m^*} = \frac{\psi}{\Delta\Pi_m}$ (fulfilling the constr.)

.... eh? what's going on? We might pay for q_{m^*} but not for the larger q_{m^*+1} ?!?

Why:

- We have assumed that it is optimal to induce effort.
- The ratio $\frac{\Delta\Pi_m}{\Pi_{m1}}$ is ~~the~~ "a strictly increasing transformation of the" -- Apr 20 likelihood ratio for effort=1

  See next slide (inserted Apr 20)
- P rewards the strongest signal that effort has been exercised — still assuming that P wants to induce effort.

The optimality-or-not of inducing effort is something else!

Likelihoods and the "monotone likelihood ratio property" (MLRP).

(MLRP defined next slide.)

* Likelihood ratio for effort:

$$\frac{Pr[e=1 \mid q_i]}{Pr[e=0 \mid q_i]} \overset{\text{Bayes}}{=} \frac{Pr[q_i \mid e=1]}{Pr[q_i \mid e=0]} \cdot \frac{Pr[e=1]}{Pr[e=0]}$$

($Pr[q_i \mid e=1] = \pi_{i1}$, $Pr[q_i \mid e=0] = \pi_{i0}$, $\frac{Pr[e=1]}{Pr[e=0]} =: k$)

* $\frac{\Delta\pi_m}{\pi_m} = 1 - 1 \Big/ \frac{\pi_{m1}}{\pi_{m0}}$ is – since k fixed – ~~[illegible]~~ strictly increasing wrt. likelihood ratio (but not = the ratio itself.)

* $\frac{\Delta\pi_m}{\pi_m}$ increasing wrt m $\Longleftrightarrow$ likelihood ratio increasing wrt m.

$\Rightarrow$ more paid for higher q

The non-monotonicity of t_m wrt m may break the model!

→ If q_{m^*+1} is produced, then deliver only q_{m^*} (if possible)

→ Non-monotonous $t(q)$ will shift the probabilities! (if is possible)

Of course, if m^* corresponds to the <u>highest</u> production level, so that this is the only rewarded delivery, this is no issue.

It is however convenient to assume a stronger property:

Def: The probabilities satisfy the <u>monotone likelihood ratio</u> property (MLRP) if

$$\frac{\Delta \pi_m}{\pi_{m1}} \text{ nondecreasing wrt } m.$$

MLRP ⇒ Can choose $t(q)$ nondecr.

MLRP ⇒ 1st o. stoch dom.

The risk averse case:

$\leadsto$ as before, transformed to concave program

$$\max_{u_0, \dots, u_{n-1}} E_1 [s - h(u)] \quad \text{s.t. inc. feas.}$$

Summing up the K-T 1st o.c.'s:

$$\pi_{m1} h'(u_m) = \lambda \cdot \Delta\pi_m + \mu \cdot \pi_{m1} \quad (*)$$

we get

$$E_1 [h'(u)] = \mu \quad \text{so } \mu > 0.$$

Further calculation

$$\leadsto \quad \lambda \psi = \operatorname{cov}(u, h'(u)) > 0 \quad (h'' > 0)$$

From (*):

xxxxxxxx
xxxxxxxx
(corr Apr 20)

$$~~t_m =~~ h'(u_m) = \lambda \frac{\Delta \pi_m}{\pi_{m1}} + \mu$$

Under MLRP, this is nondecr. wrt m

Now n+2 eq's (nonlinear!)

in n var's + 2 multipliers.

Signals

* We observe a (verifiable!) signal $\in \{\sigma_0, \sigma_1\}$, where σ_1 is more likely to occur if effort = 1
  → we disregard "equally likely" ⟺ noninformative signal.
* The signal may be part of contract
  → and if so: <u>will</u> reduce P's expected cost
    → except ~~ ~~ risk neutral A.
* How?

[* Signal drawn conditionally independent of q.]

Example:

→ Lånekassen rewards passing an exam

→ Lånekassen wants effort. (Assume MLRP.)

→ Lånekassen could ask for signal:
"termpaper submitted?"

→ ... hoping that "yes" is more likely if effort.

→ ~~model illustration:~~ (cartoon. Omitted)

Assume: "good news"

$$\Pr[\text{signal} = \sigma_1 \mid e = i] = \nu_i,$$

so that $\nu_1 > \nu_0$.

Two possible outputs q_0, q_1

× Two possible signals σ_0, σ_1

Four possible pairs a_j

"1" ⇔ a_1
"0" ⇔ $e=0$

For $a_1 = (q_1, \sigma_1)$: (_does_ actually correspond to the book's pair #1 -- cf. my confusion @ the lecture)

If $e=0$, then $\Pr[a_1 \mid e=0] = \pi_0 \nu_0 = P_{10}$

If $e=1$: $\Pr[a_1 \mid e=1] = \pi_1 \nu_1 = P_{11}$

Turns out optimal for $h'(u_1)$ ← "1" now for a_1

not $\mu + \lambda\left(1 - \frac{\pi_{m0}}{\pi_{m1}}\right)$ ← m as in q_m

but $\mu + \lambda\left(1 - \frac{P_{j0}}{P_{j1}}\right)$ ← j as in a_j

revised slide

The book's table 4.1 p 168
& formulas (4.57) – (4.60)

E.g. RHS of (4.59):

$$\mu + \lambda \underbrace{\left(\frac{(1-\pi_1)\nu_1 - (1-\pi_0)\nu_0}{(1-\pi_1)\nu_1} \right)}_{}$$

" $\frac{\Delta P}{P}$ " notation.

Equals:

$$1 - \frac{(1-\pi_0)\nu_0}{(1-\pi_1)\nu_1}$$

$$= 1 - \frac{P_{j0}}{P_{j1}} \qquad \left(\begin{matrix} j=3 \text{ in} \\ \text{table 4.1} \end{matrix} \right)$$

Table 4.1: don't memorize – copy the thinking

State of nature				The interesting quantity:
γ_1	P_{10}	… …	P_{11}	P_{10} / P_{11}
γ_2	⋮		⋮	⋮
γ_3				
γ_4	P_{40}	…	P_{41}	P_{40} / P_{41}