Intertemporal macroeconomics

Econ 4310 Lecture 4. Part 1

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Ramsey-model with underlying natural growth

$$\max\ U_0 = \sum_{t=0}^{\infty} \beta^t u(c_t A_t) L_t \tag{1}$$

$$\text{given } c_t = k_t + f(k_t) - (1+n)(1+g)k_{t+1},\ k_0 = \bar{k_0}, \tag{2}$$

$$A_t = A_0(1+g)^t,\ L_t = L_0(1+n)^t$$

$$\text{and } k_t \geq 0,\ c_t \geq 0$$

$$0 < \beta = 1/(1+\rho) < 1$$

Maximization is with respect to c_t and k_{t+1} for $t = 0, 1, 2, \ldots$

The value function

Definition:

$$V(k_t, A_t, L_t) = \max \sum_{j=0}^{\infty} \beta^t u(c_j A_j) L_j \tag{3}$$

Bellman equation:

$$V(k_t, A_t, L_t) = \max_{c_t, k_{t+1}} [u(c_t A_t) L_t + \beta V(k_{t+1}, A_{t+1}, L_{t+1})] \tag{4}$$

Maximization is subject to constraint (2) etc. Insertion yields:

$$V(k_t, A_t, L_t) =$$
$$\max_{k_{t+1}} [u([k_t + f(k_t) - (1+n)(1+g)k_{t+1}]A_t)L_t + \beta V(k_{t+1}, A_{t+1}, L_{t+1})]$$

Deriving first order conditions

$$\max_{k_{t+1}} [u([k_t + f(k_t) - (1+n)(1+g)k_{t+1}]A_t)L_t + \beta V(k_{t+1}, A_{t+1}, L_{t+1})]$$

Differentiation yields

$$u'(c_t A_t)[-(1+n)(1+g)]A_t L_t + \beta V_1'(k_{t+1}, A_{t+1}, L_{t+1}) = 0$$

Since $V_1'(k_{t+1}, A_{t+1}, L_{t+1})$ is the result of maximization,

$$V_1'(k_{t+1}, A_{t+1}, L_{t+1}) = u'(c_{t+1}A_{t+1})A_{t+1}L_{t+1}(1 + f'(k_{t+1}))$$

Combining the two above yields

$$u'(c_t A_t)[-(1+n)(1+g)]A_t L_t + \beta u'(c_{t+1}A_{t+1})A_{t+1}L_{t+1}(1 + f'(k_{t+1})) = 0$$

which simplifies to

$$u'(c_t A_t) = \beta u'(c_{t+1}A_{t+1})(1 + f'(k_{t+1})) \tag{5}$$

CES-preferences

$$u(x) = (1/(1-\theta))x^{1-\theta}, \sigma = 1/\theta > 0$$

First order condition:

$$(c_t A_t)^{-\theta} = (c_{t+1} A_{t+1})^{-\theta} \beta(1 + f'(k_{t+1}))$$

$$\frac{(c_t A_t)^{-\theta}}{(c_{t+1} A_{t+1})^{-\theta}} = \beta(1 + f'(k_{t+1})) \tag{6}$$

Growth rate for consumption per capita:

$$\frac{c_{t+1} A_{t+1}}{c_t A_t} = [\beta(1 + f'(k_{t+1}))]^{\sigma}$$

Conditions for balanced growth

Growth in consumption per efficiency unit of labor:

$$\begin{aligned} \frac{c_{t+1}}{c_t} &= \frac{1}{1+g}[\beta(1+f'(k_{t+1}))]^{\sigma} \\ &= (1+g)^{-1}(1+\rho)^{-\sigma}[(1+f'(k_{t+1}))]^{\sigma} \end{aligned} \tag{7}$$

Balanced growth $(k_{t+1} = k_t = k^*, c_{t+1} = c_t = c^*)$ requires:

$$1 = (1+g)^{-1}(1+\rho)^{-\sigma}[(1+f'(k^*))]^{\sigma} \tag{8}$$

$$c^* = f(k^*) - (n+g+ng)k^* \tag{9}$$

Loglinearizing steady state condition (8)

$$(1+g)^{-1}(1+\rho)^{-\sigma}[(1+f'(k^*))]^{\sigma} = 1$$

$$-\ln(1+g) - \sigma\ln(1+\rho) + \sigma\ln(1+f'(k^*)) = \ln 1 = 0$$

$$-g - \sigma\rho + \sigma f'(k^*) \approx 0$$

$$f'(k^*) \approx \rho + \frac{1}{\sigma}g \tag{10}$$

Short period, g, ρ and $f'(k)$ small numbers, $ln(1+x) \approx x$

Observations on the steady state

$$f'(k^*) \approx \rho + \frac{1}{\sigma} g$$

- k^* is independent of n
- k^* depends negatively on ρ
- k^* depends negatively on g and more so the lower is σ
- k^* depends positively on σ when $g > 0$

High g implies high real interest rate.

Comparison of steady states

$$\text{Golden rule } f'(k^{**}) \approx n + g$$

$$\text{Ramsey } f'(k^{*}) \approx \rho + \frac{1}{\sigma} g$$

$$k^{*} < k^{**} \iff f'(k^{*}) > f'(k^{**}) \iff \rho + \frac{1}{\sigma} g > n + g$$

$$k^{*} < k^{**} \iff \rho > (1 - \theta) g + n$$

How come that it is not satisfied always? $k^{*} > k^{**}$ is impossible on an optimal path.

Value of objective function infinite when $\rho < (1-\theta)g + n$

$$\sum_{t=0}^{\infty} \beta^t \frac{1}{\theta}[c^* A_0 (1+g)^t]^{1-\theta} L_0 (1+n)^t = \sum_{t=0}^{\infty} \text{Const.}(\beta(1+g)^{1-\theta}(1+n))^t$$

Diverges if

$$= (1+\rho)^{-1}(1+g)^{1-\theta}(1+n) > 1$$

$$-\ln(1+\rho) + (1-\theta)\ln(1+g) + \ln(1+n) > 0$$

$$\rho < (1-\theta)g + n$$

Same as condition for $k^* > k^{**}$. Optimization not meaningful when $\rho < (1-\theta)g + n$.