Partial solution of PS3, question 2

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To make sure you all get the details right on the OLG model in PS3 and correct some mistakes from the seminar, here is the corrected fromalism of this question:

1. We normalize by dividing by the number of efficient workers $\hat{L}_t = L_t\gamma^t$ so

$$\begin{aligned} k_t &= \frac{K_t}{\hat{L}_t} \\ y_t &= \frac{K_t}{\hat{L}_t} = e^{z_t}k_t^{\alpha} \end{aligned}$$

As long as z_t is stationary, this yields stationary variables.

3. The consumption allocation solves

$$\max_{c_{1t}} \ln\left(c_{1t}\right) + \ln\left[\left(1+r_{t+1}\right)\left(w_t - c_{1t}\right)\right]$$

which yields the first order condition

$$\frac{1}{c_{1t}} = \frac{1}{w_t - c_{1t}} \Longrightarrow c_{1t} = s_{1t} = \frac{w_t}{2}.$$

4. The wage rate per efficient worker is

$$\bar{w}_t = \frac{\partial Y_t}{\partial \hat{L}_t} = \left(1-\alpha\right)e^{z_t}k_t^{\alpha},$$

yielding a wage per worker of

$$w_t = \gamma^t w_t = \left(1-\alpha\right)\gamma^t e^{z_t}k_t^{\alpha}.$$

Similarly the rental rate of capital is[1]

$$r_t = \frac{\partial Y_t}{\partial K_t} = \alpha e^{z_t}k_t^{\alpha-1}$$

[1]The correct way to think about this rate is that the old rent out their savings to the firms, get a rental rate of r_t and their savings back, and end their life by consuming both.

To find the evolution of capital, we know that in aggregate terms,

$$K_{t+1} = L_t s_{1t},$$

Hence in normalized terms, we get

$$k_{t+1} = \frac{K_{t+1}}{\hat{L}_{t+1}} = \frac{L_t s_{1t}}{\hat{L}_{t+1}} = \frac{L_t \frac{1}{2} (1-\alpha) \gamma^t e^{z_t} k_t^\alpha}{(1+n) \gamma^{t+1} L_t}$$

so[2]

$$k_{t+1} (1+n) \gamma = \frac{1-\alpha}{2} e^{z_t} k_t^\alpha.$$

The steady state level of capital k^* solves

$$k^* (1+n) \gamma = \frac{1-\alpha}{2} e^{z_t} k^{*\alpha} \Longrightarrow k^* = \left(\frac{1-\alpha}{2 (1+n) \gamma} \right)^{\frac{1}{1-\alpha}}$$

[2]Notice that the growth rate of productivity is $g = \gamma - 1$. We can then use the approximation $(1+n)\gamma \simeq 1 + n + g$ as both n and g are small numbers so ng is very small.