# ECON 4310 Problem set 5
# Due Nov. 10

**1. Precautionary saving**

Consider an economy with constant population, $N(t) = N$ where agents have the following utility function:

$$u_t^h = \log(c_t^h(t) + \log(c_t^h(t+1))$$

The endowment of young agents is always $\omega t(t) = 1$, while the endowment of an old agents is uncertain - half of the old get $\omega_t^h(t+1) = 1/2$ and the others get $\omega_t^h(t+1) = 3/2$, and an individual's outcome is not known at birth:

$$\omega_t(t+1) = \begin{cases} 1/2 & \text{with } 50\% \text{ probability} \\ 3/2 & \text{with } 50\% \text{ probability} \end{cases}$$

There is only one asset - private lending - and no insurance against the endowment risk. Note that there is no aggregate risk (given that there are, say, equally many with high and low second-period endowment).

i. Write down the first order condition of the agents in order to solve their optimal savings problem.

ii. Can you express the equilibrium conditions for this economy?

iii. Compute the competitive equilibrium and the interest rate of the economy.

iv. Explain why agents would like to buy insurance against the endowment risk when old.

**2. Computation**

Consider a model economy where the social planner chooses an infinite sequence of consumption and next period's capital stock $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ in order to

$$\max_{\{ct,kt+1\}_{t=0}^{\infty}} E_t \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$\begin{aligned} c_t + k_{t+1} \leq y_t + (1-\delta)k_t, & \qquad \forall t \\ ct, kt \geq 0, & \qquad \forall t \\ k0 > 0 & \qquad \text{given.} \end{aligned}$$

The model economy is exposed to an iid stochastic shock in each period,

$$\gamma_t \in \Gamma = [4.95, 5.05]$$

with associated probabilities

$$\pi_1 = Pr(\gamma_t = \gamma_1) = .5$$
$$\pi_2 = Pr(\gamma_t = \gamma_2) = .5$$

Assume the following functional forms

$$\begin{aligned} u(c_t) = \frac{c_t^{1-\sigma} - 1}{1-\sigma} & \qquad \forall t, \sigma > 0 \\ y_t = \gamma_t k_t^{\alpha}, \forall t, & \qquad \alpha \in (0,1) \end{aligned}$$

Compute the value function and the decision rules to this deterministic problem using Bellman's method of successive iterations, also called value function iterations:

i. Reformulate this problem as a dynamic programming problem, i.e. write up the Bellman equation. What are the control variable(s) and the (endogenous and exogenous) state variable( s)?

ii. Compute the non-stochastic steady state value of the capital stock $k^*$, i.e. use the mean value of $\gamma$.

For these computations, set $\alpha = .35, \beta = .98, \delta = .025, \sigma = 2, \gamma = 5$.

iii. In Matlab, make a discretization of the state space by constructing a grid on the capital stock with $g$ values $k \in X = [k_1 < k_2 < \ldots < k_g]$ with $k_1 > 0$ (i.e. construct the vector $X$).

Hint: The command `linspace` may be useful.

iv. Construct consumption and welfare matrices. Compute the two $(g \times g)$-dimensional consumption matrix $C1$ and $C2$, conditional on the value of the stochastic shock, with the value of consumption for all the $(g \times g)$ values of $k$ and $k'$ (where $k'$ is next period's capital stock. Next compute two $(g \times g)$-dimensional matrices with the utility of consumption for all the $(g \times g)$ values of $k$ and $k'$.

v. Define the initial value function and compute the first period value function: Define an initial $(g \times 1)$-dimensional vector (of zeros) for the initial value function $v_0$. Compute the two $(g \times 1)$-dimensional vectors of value functions from the first step of the value function iteration as

$$v_{11}(k, \gamma_1) = \max_{k' \in X} \{U_1 + \beta E v_0(k', \gamma',$$

$$v_{21}(k, \gamma_2) = \max_{k' \in X} \{U_1 + \beta E v_0(k', \gamma'$$

vi. Continue by iterating on Bellman's equation until convergence whereby we have computed a close approximated true value function by solving Bellman's equation.

vii. Compute the approximated true decision rules, $k_0 = g(k)$ and $c = Ak^\alpha - k'$, based on the approximated true value function by solving Bellman's equation.

(a) Maximize the final, approximated value functions and find the index row number, $j$, where the maximized value for each $k$ for each of the two functions.

(b) Compute the decision rules for capital from the index, $j$, $k' = k(j)$. Compute the decision rule for consumption residually.