Labor supply in RBC models + calibration
Lecture 12, ECON 4310

Tord Krogh

October 8, 2012
Summary from last lecture

Last lecture (#11) we went through

- What a solution is for our basic model
- How to linearize the deterministic model
- How to linearize the stochastic model
We saw in the deterministic case that the conditions describing optimum:

\[ u'(c_t) = \beta [1 - \delta + \alpha A k_{t+1}^{\alpha-1}] u'(c_{t+1}) \]
\[ c_t + k_{t+1} = A k_t^\alpha + (1 - \delta) k_t \]

could be linearized around steady state as:

\[ \hat{c}_t = \hat{c}_{t+1} + \beta \frac{1}{\theta} (1 - \alpha) A k^* k_{t+1}^{\alpha-1} \hat{k}_{t+1} \]
\[ c^* \hat{c}_t = \frac{1}{\beta} k^* \hat{k}_t - k^* \hat{k}_{t+1} \]
The conditions can be used to find
\[ \hat{k}_{t+1} = a_2 \hat{k}_t \]
and
\[ \hat{c}_t = \frac{k^*}{c^*} (\beta - a_2) \hat{k}_t \]
which is our solution to the model for an initial value of \( \hat{k}_0 \). The solution for the stochastic case is more complicated because of expectations, but very similar.
Correction: With \( \hat{x}_t = (x_t - x^*)/x^* \), I wrote last lecture that
\[
x_t \approx x^*(1 + \hat{x}_t)
\]
This is of course not an approximation. The correct is:
\[
x_t = x^*(1 + \hat{x}_t)
\]
Reason for confusion? Sometimes \( \hat{x} \) is defined as the log-deviation \( (\log x_t - \log x^*) \), and in that case it is an approximation.
Today’s lecture

- Introducing labor supply in the basic model
  - The *intratemporal* optimality condition
  - Frisch elasticity and IES for labor supply
- Labor lotteries
- The concept of calibration
Labor supply in the basic model

So far in the course we have considered models where a representative agent (or a social planner) maximizes

\[ \sum_{t=0}^{\infty} \beta^t u(c_t) \]

with some fixed amount of labor available for production. Now we consider the more general case where we maximize

\[ \sum_{t=0}^{\infty} \beta^t U(c_t, h_t) \]

with \( h_t \) measuring hours worked, making \( 1 - h_t \) the hours of leisure.
In RBC models we will see that the labor supply response to changes in wages (driven by productivity shocks) is an important propagation mechanism.
To understand the basics, take one step back, and consider only a simple two-period model of labor supply, where we assume that utility is separable in consumption and labor supply:

\[
\max_{\{c_0, c_1, h_0, h_1, a_1\}} \left( u(c_0) - v(h_0) + \beta [u(c_1) - v(h_1)] \right)
\]

\[\text{s.t.}\]
\[c_0 + a_1 = w_0 h_0 + (1 + r_0) a_0\]
\[c_1 = w_1 h_1 + (1 + r_1) a_1\]

for \(a_0\) given.
Labor supply in the basic model IV

This problem has the following first order conditions (letting $\lambda_0$ and $\lambda_1$ be the Lagrange multipliers)

1. $u'(c_0) = \lambda_0$
2. $\beta u'(c_1) = \lambda_1$
3. $\nu'(h_0) = \lambda_0 w_0$
4. $\beta \nu'(h_1) = \lambda_1 w_1$
5. $\lambda_0 = \lambda_1(1 + r_1)$
Labor supply in the basic model V

- As before, combine (1), (2) and (3) to find the Euler equation:

\[ u'(c_0) = \beta (1 + r_{t+1}) u'(c_1) \]

We refer to the Euler equation as the *intertemporal optimality condition*.

- Then to learn more about labor supply, combine (1) and (3) to find:

\[ \frac{v'(h_0)}{u'(c_0)} = w_0 \]

This is a standard MRS = relative price condition. The LHS measures the utility loss (in terms of $c_0$) of one extra hour of work. The RHS gives the gain (in terms of $c_0$) from taking this hour of leisure. We refer to this as the *intratemporal optimality condition*.

- A similar condition holds of course for the last period:

\[ \frac{v'(h_1)}{u'(c_1)} = w_1 \]
Notice that you can combine the Euler equation with the intratemporal optimality conditions to find:

\[ \frac{v'(h_0)}{w_0} = \beta (1 + r_1) \frac{v'(h_1)}{w_1} \]

or:

\[ \frac{\beta v'(h_1)}{v'(h_0)} = \frac{w_1}{(1 + r_1)w_0} \]

which we can refer to as the *intertemporal* labor supply condition. It is illustrating that we also face a choice along the intertemporal dimension when we choose labor supply.
OK. Summary? We have one Euler equation and two intratemporal conditions:

\[ u'(c_0) = \beta (1 + r_1) u'(c_1) \]
\[ \nu'(h_0) = u'(c_0) w_0 \]
\[ \nu'(h_1) = u'(c_1) w_1 \]

These three equations, together with the resource constraints:

\[ c_0 + a_1 = w_0 h_0 + (1 + r_0) a_0 \]
\[ c_1 = w_1 h_1 + (1 + r_1) a_1 \]

will determine the five endogenous variables \( c_0, c_1, h_0, h_1 \) and \( a_1 \).
Assume that

\[ u(c) - v(h) = \log c - \phi \frac{h^{1+\theta}}{1 + \theta} \]

The Euler equation and the intratemporal conditions are in this case given by:

\[ c_1 = \beta (1 + r_1) c_0 \]

\[ \phi h_0^\theta = \frac{w_0}{c_0} \]

\[ \phi h_1^\theta = \frac{w_1}{c_1} \]
As we have seen before when utility of consumption is a log-function, we can combine the Euler equation with the resource constraints to find

\[ c_0 = \frac{1}{1 + \beta} \left[ w_0 h_0 + \frac{w_1 h_1}{1 + r_1} \right] \]

This solution for \( c_0 \), together with

\[ \phi h_0^\theta = \frac{w_0}{c_0} \]
\[ \phi h_1^\theta = \frac{w_1}{\beta(1 + r_1)c_0} \]

are the conditions for optimum.
Combining the intratemporal conditions we find

\[
\left( \frac{h_1}{h_0} \right)^\theta = \frac{w_1}{\beta (1 + r_1) w_0}
\]

or

\[
h_1 = \left( \frac{w_1}{\beta (1 + r_1) w_0} \right)^{\frac{1}{\theta}} h_0
\]
Labor supply in the basic model X

Then solve for $h_0$ by using the expressions for $c_0$ and $h_1$:

$$\phi h_0^{\theta} = \frac{w_0}{c_0}$$

$$\Rightarrow \phi h_0^{\theta} \left[ w_0 h_0 + \frac{w_1 h_1}{1 + r_1} \right] = w_0 (1 + \beta)$$

$$\Rightarrow \phi h_0^{\theta} \left[ w_0 h_0 + \frac{w_1}{1 + r_1} \left( \frac{w_1}{\beta (1 + r_1) w_0} \right)^{\frac{1}{\theta}} h_0 \right] = w_0 (1 + \beta)$$

$$\Rightarrow \phi h_0^{\theta} \left[ h_0 + \frac{w_1}{(1 + r_1) w_0} \left( \frac{w_1}{\beta (1 + r_1) w_0} \right)^{\frac{1}{\theta}} h_0 \right] = (1 + \beta)$$

$$\Rightarrow \phi h_0^{1+\theta} \left[ 1 + \left( \frac{w_1}{(1 + r_1) w_0} \right)^{1+\frac{1}{\theta}} \beta^{-\frac{1}{\theta}} \right] = (1 + \beta)$$
Labor supply in the basic model XI

What is there to learn from this equation?

\[ \phi h_0^{1+\theta} \left[ 1 + \left( \frac{w_1}{(1 + r_1)w_0} \right)^{1+\frac{1}{\theta}} \beta^{-\frac{1}{\theta}} \right] = (1 + \beta) \]

- $h_0$ is increasing in $w_0$
- But it is also decreasing in $w_1$ (intertemporal substitution)
- An increase in $w_0$ and $w_1$ of the same relative size will not affect labor supply!
- So you get the result that if only $w_0$ goes up, then $h_0$ is also increased. But if $w_0$ and $w_1$ go up with $w_1/w_0$ constant, $h_0$ is unchanged. And if $w_1$ goes up, $h_0$ goes down.

[These conclusions are of course dependent on the utility function you use, but they illustrate general tendencies]
Two important elasticities

There are two important elasticities we need to care about:

1. **Frisch elasticity**: The elasticity of labor supply with respect to the wage, keeping marginal utility of wealth constant. *Measures the substitution effect*

2. **Intertemporal elasticity of substitution (IES) for labor supply**: The elasticity of relative labor supply *across* periods with respect to the present value of wage growth
Frisch elasticity

How to find the Frisch elasticity? Use the intratemporal optimality condition.

\[
\frac{v'(h_t)}{u'(c_t)} = w_t
\]

for \( t = 0, 1 \). For a given marginal utility of consumption, this defines an implicit function \( h_t = q(w_t) \). Let us differentiate with respect to \( w_t \):

\[
\frac{v''(q(w_t))q'(w_t)}{u'(c_t)} = 1
\]
Frisch elasticity II

Then we multiply by $v'(q(w_t))/q(w_t)$:

$$\frac{v'(q(w_t))}{q(w_t)} \frac{v''(q(w_t))q'(w_t)}{u'(c_t)} = \frac{v'(q(w_t))}{q(w_t)}$$

Divide both sides by $v''(q(w_t))$ and re-arrange the terms on the left to get

$$El_{w_t} h_t = El_{w_t} q(w_t) = \frac{w_t}{q(w_t)} q'(w_t) = \frac{v'(h_t)}{h_t v''(h_t)}$$

This is the Frisch elasticity of labor supply.
Continue using our last choice for $v(h)$:

$$v(h_t) = \phi \frac{h_t^{1+\theta}}{1 + \theta}$$

With this, $v'(h) = \phi h_t^\theta$ and $v''(h) = \theta \phi h_t^{\theta-1}$, implying:

$$El_{w_t} h_t = \frac{\phi h_t^\theta}{h_t \theta \phi h_t^{\theta-1}} = \frac{1}{\theta}$$

i.e. a constant Frisch elasticity at $1/\theta$. 
IES for labor supply

What about the IES for labor supply? Keep the particular choice of $\nu(h)$. To find this elasticity, we use the *inter*temporal optimality condition for labor:

$$
\frac{\beta \nu'(h_1)}{\nu'(h_0)} = \frac{w_1}{(1 + r_1)w_0}
$$

which now becomes

$$
\beta \left( \frac{h_1}{h_0} \right)^\theta = \frac{w_1}{(1 + r_1)w_0} = \tilde{W}_0
$$

where $\tilde{W}_1$ denotes the present value of wage growth.
IES for labor supply II

The IES for labor supply is the elasticity of $h_1/h_0$ with respect to $\tilde{W}_0$. To find it, we can either find derivatives etc. like for the Frisch case, or simply use that:

$$E_{l_x}y = \frac{d \log y}{d \log x}$$

Taking logs of the intertemporal optimality condition for labor we get:

$$\log \beta + \theta \log \left( \frac{h_1}{h_0} \right) = \log \tilde{W}_0$$

Hence:

$$E_{l_{\tilde{W}_0}} \frac{h_1}{h_0} = \frac{1}{\theta}$$

In this case the IES for labor supply equals the Frisch elasticity.
Using the elasticities

- The higher the Frisch elasticity, the more willing are you to work if the wage increases.
- The higher the IES for labor supply, the more willing are you to shift the *path* of labor supply in response to temporary changes in the wage.
Elasticities

Using the elasticities II

With $v(h) = \phi \frac{h^{1+\theta}}{1+\theta}$:

- Empirical estimates of the Frisch elasticity are often in the range of 0.5, implying $\theta = 2$.
- In contrast, maximum volatility in hours is obtained by setting $\theta = 0$ (since then the Frisch elasticity $\rightarrow \infty$). This would make

$$v(h) = \phi h$$

i.e. linear in hours.
Since we want to choose values for our structural parameters that are consistent with micro evidence, we should also set $\theta$ close to 2 in an RBC model.

But values of $\theta$ around 2 are often producing too little volatility in labor supply in RBC models!

To get more volatile labor supply, one would rather be somewhere closer to $\theta = 0$, in which case $v(h)$ is linear in $h$ and we get maximum volatility.

This is a problem

But we know that (e.g. as shown in Kydland and Prescott, 1990) fluctuations in labor supply seems to be driven primarily by changes in the extensive margin – not so much by the intensive. Can we change our model to account for this?
This is the motivation for models of *indivisible* labor combined with labor lotteries (see Hansen (1984) and Rogerson (1988)).

- In the simple model the agent could choose $h$ to be anywhere between zero and one
- With indivisible labor, we will require $h = \{0, 1\}$, i.e. working becomes a ‘yes/no’ choice
- Labor lotteries (Rogerson, 1988) offers an elegant way of introducing this mechanism
Consider the following setting:

- There exists a continuum of households on the unit interval, each with a utility function
  \[ \sum_{t=0}^{\infty} \beta^t [u(c_t) - v(h_t)] \]
- Hours worked must by each agent is either 0 or 1
- All agents agree to join in a ‘labor lottery’: With probability \( \xi_t \) they will have to work, and with probability \( 1 - \xi_t \) they will be unemployed. But no matter if they work or not, all will receive the same income (and therefore consumption).
- \( \xi_t \) is then chosen by the group or a social planner to maximize welfare
- With a continuum of agents, \( \xi_t \) can be interpreted as the share of agents that must work
Labor lotteries III

Since all agents are the same, we maximize welfare by maximizing

$$E \left\{ \sum_{t=0}^{\infty} \beta^t [u(c_t) - v(h_t)] \right\} = E \left\{ \sum_{t=0}^{\infty} \beta^t [u(c_t) - v(h_t)] | \text{Work} \right\}$$

$$+ E \left\{ \sum_{t=0}^{\infty} \beta^t [u(c_t) - v(h_t)] | \text{Not work} \right\}$$

$$= \sum_{t=0}^{\infty} \xi_t \beta^t [u(c_t) - v(1)] + \sum_{t=0}^{\infty} (1 - \xi_t) \beta^t [u(c_t) - v(0)]$$

$$= \sum_{t=0}^{\infty} \beta^t [u(c_t) - \xi_t v(1) - (1 - \xi_t) v(0)]$$

$$= \sum_{t=0}^{\infty} \beta^t [u(c_t) - \xi_t [v(1) - v(0)] - v(0)]$$

Let us define \( D = v(1) - v(0) \) and ignore the last \( v(0) \) term (since a constant is not relevant for maximizing a function). The objective function we are left with is

$$\sum_{t=0}^{\infty} \beta^t [u(c_t) - D \xi_t]$$
Labor lotteries IV

But this is like magic! We started out with an economy where every agent was identical, such that the social planner problem would be to maximize

$$
\sum_{t=0}^{\infty} \beta^t [u(c_t) - v(h_t)]
$$

Introducing labor lotteries instead, gives us:

$$
\sum_{t=0}^{\infty} \beta^t [u(c_t) - D\xi_t]
$$

where $\xi_t$ can be interpreted as our new ‘labor supply’ since total labor supply $n_t$ must equal $\xi_t$. This latter utility function is linear in labor supply, which gives us hope that it will also give larger labor supply responses when shocks are hitting the economy.
Recall, if we have

$$v(h) = \phi \frac{h^{1+\theta}}{1 + \theta}$$

then $\frac{1}{\theta}$ is the Frisch elasticity.

We can set $\theta = 2$ to have micro elasticities that are plausible.

For the model with labor lotteries, the value of $\theta$ only affects $D$, since:

$$D = v(1) - v(0) = \phi \frac{1}{1 + \theta}$$

so it does not affect the substitution effects.

Since the labor lotteries model gives us a model as if utility was linear, we get a macro Frisch elasticity equal to infinity, no matter what we set the micro elasticity to be!

So there is a difference between micro and macro elasticities.
Intuition for the possible difference between micro and macro elasticities:

- For the micro elasticity, we look at the effect on hours worked from a marginal change in the wage. When hours are changing, your disutility of labor change as well, dampening the impact.

- For a macro elasticity, we only look at the effect on aggregate hours worked when the wage level changes. If all labor is indivisible, all changes in ours are due to people going from unemployment to employment. Their disutility of work is constant since work is a zero-one choice. So there is no dampening effect from changes in disutility of labor.
RBC models therefore often assume utility functions where utility is linear in labor supply, using a labor lottery argument as fundament.
Our basic model combined with a labor lottery assumption gives then the following social planner’s problem:

\[
\max \quad \{c_t, h_t, k_{t+1}\}_{t=0}^{\infty} \sum_{t=0}^{\infty} \beta^t [u(c_t) - Dn_t] \\
\text{s.t.} \\
c_t + k_{t+1} = Ak_t^\alpha n_t^{1-\alpha} + (1 - \delta)k_t \\
c_t \geq 0 \\
k_{t+1} \geq 0 \\
0 \leq n_t \leq 1
\]

with \( k_t > 0 \) given. We continue to ‘ignore’ the conditions of \( c, k \) and \( n \), since we will find an interior solution.
Form the Lagrangian as before ($\lambda_t$ being the Lagrange multiplier), and find the first-order conditions.

- With respect to $c_t$:
  $$\beta^t u'(c_t) = \lambda_t$$  \hspace{1cm} (6)

- With respect to $n_t$:
  $$D = \lambda_t A (1 - \alpha) k_t^\alpha n_t^{-\alpha}$$  \hspace{1cm} (7)

- With respect to $k_{t+1}$:
  $$\lambda_t = \lambda_{t+1} [A \alpha k_{t+1}^{\alpha-1} n_t^{1-\alpha} + 1 - \delta]$$  \hspace{1cm} (8)
As before, combine (1) and (3) to find the Euler equation:

$$u'(c_t) = \beta (1 + r_{t+1}) u'(c_{t+1})$$

where $r_{t+1} = A\alpha k_{t+1}^{\alpha-1} n_t^{1-\alpha} - \delta$.

Combine (1) and (2) to find the intratemporal optimality condition:

$$\frac{D}{u'(c_t)} = w_t$$

where $w_t = A(1 - \alpha) k_t^\alpha n_t^{-\alpha}$
With $n$ fixed (before today), optimum required the following conditions to be satisfied:

- The Euler equation
- The resource constraint

Introducing labor supply and making $n$ be set optimally adds one extra restriction:

- The intratemporal optimality condition
Next steps? Like in Lecture 11:
- Characterize steady state
- Linearize conditions around steady state
- Solve the set of linearized equations
- Plot impulse-response functions, simulate, calculate moments etc.

We can save this to next lecture.
One thing we will not save to next lecture is: **How should we choose values for the structural parameters in an RBC model?** What is most frequently applied is called *calibration*.
Calibration II

Take the basic model with labor lottery. Assume that the utility function is

\[ u(c) = \log c \]

Ignoring productivity, the model has four structural parameters:

- Discount factor \( \beta \)
- Depreciation rate \( \delta \)
- Cobb-Douglas parameter \( \alpha \)
- Disutility of labor supply \( D \)

To calibrate the model we must find four *moments* (usually averages) we want our model to match. By this we mean that the *steady state* properties of the model should match the data.
A standard set of moments to match are:

- Average capital share of income
- Average investment to capital ratio
- Average long-term real interest rate
- Average share of available hours spent on work

Let us see how we can use each of these moments to calibrate our model.
Start with the average capital share. Say that we have observed an average US capital share of 1/3 over the last 50 years. To use this fact, let us calculate what the capital share in our model is:

\[
\frac{r_t k_t}{y_t} = \frac{\alpha A k_t^{\alpha - 1} n_t^{1-\alpha} k_t}{A k_t^\alpha n_t^{1-\alpha}} = \alpha
\]

So if we set \( \alpha = 1/3 \), we ensure that the model implies a realistic capital share.
Then take the average investment to capital ratio. Usually we only observe $\frac{I}{Y}$ and $\frac{K}{Y}$. Say that we’ve calculated an average investment to output share of 0.25 and capital to output share of 10. To use this fact, let us look at the law of motion for capital

$$k_{t+1} = (1 - \delta)k_t + i_t$$

Divide by output and use that $\frac{k_t}{y_t}$ is constant in steady state. That gives us:

$$\delta = \frac{i}{k} = \frac{i}{y} \left( \frac{k}{y} \right)^{-1}$$

So if we set $\delta = 0.025$, we ensure that the model implies a realistic investment to capital ratio in steady state.
Then there is the long-term interest rate. If our model is quarterly, it could be that 1% real interest rate is realistic. The Euler equation in steady state (constant consumption) gives us:

\[ 1 = \beta (1 + r) \]

or

\[ \beta = \frac{1}{1 + r} \]

So if we set \( \beta = 1/1.01 \approx 0.99 \), we ensure that the model implies a realistic real interest rate in steady state.
Finally: the share of hours available that is spent on work. Maybe $n = 1/3$ is realistic. Use the intratemporal optimality condition in steady state:

$$\frac{D}{u'(c)} = w$$

When $u(c) = \log c$ this can be written as

$$D = \frac{w}{c} = \frac{1}{n} \frac{wn}{c} = \frac{1}{n} \frac{(1-\alpha)y}{c} = \frac{1}{n} \frac{1-\alpha}{c/y} = \frac{1}{n} \frac{1-\alpha}{1 - i/y}$$

With $n = 1/3$, $\alpha = 1/3$ and $i/y = 0.25$, this gives

$$D = \frac{1}{1/3} \frac{1 - 1/3}{1 - 0.25} = \frac{8}{3}$$

So if we set $D = 8/3$, we ensure that the model implies a realistic share of hours spent on work.
Summary? If we want to ensure a capital share equal to 1/3, an investment to capital ratio of 2.5%, a real interest rate of 1% and $n = 1/3$ in our model we just choose:

- $\alpha = 1/3$
- $\delta = 0.025$
- $\beta = 1/1.01$
- $D = 8/3$
Calibrating the model in this way ensures that the model has reasonable *long-run* properties.

So it is not impressing that our RBC model manages to replicate these facts.

The interesting question is: How well will a simple model calibrated to match long-run facts do when it comes to explain business cycles?
Some more on calibration of labor supply

What does $D = 8/3$ imply for the parameters in $v(h)$? We keep on assuming

$$v(h) = \phi \frac{h^{1+\theta}}{1 + \theta}$$

so that $D = \phi/(1 + \theta)$. This shows that if we want $\theta = 2$ (to be consistent with micro data), we need $\phi = 8$. 

Then imagine that we were back to the model with \textit{divisible} labor. In that model the intratemporal optimality condition in steady state is

\[
\frac{\nu'(h)}{u'(c)} = w
\]

or with $\nu(h)$ as specified and log utility:

\[
\phi h^\theta = \frac{w}{c}
\]

Doing the same transformations on the RHS as earlier, we get

\[
\phi h^\theta = \frac{1}{h} \frac{1-\alpha}{1-i/y}
\]

which gives

\[
\phi = \frac{1}{h^{1+\theta}} \frac{1-\alpha}{1-i/y}
\]

If $\theta = 2$ (and the remaining calibration is as before), we have $\phi = 24$. 
So we could of course also calibrate a model with divisible labor to obtain $h = 1/3$ in steady state. The effect is an implicit selection of a much larger value of $\phi$.

But that does not change the main difference between the labor lottery and divisible labor models: The difference in substitution effects!
Three things you MUST remember from today

1. What is the Frisch elasticity?
2. Why do we use the labor lotteries model?
3. How do we choose values for the structural parameters in an RBC model?