Lecture notes for 4310
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1 Introduction to overlapping generations and Ricardian Equivalence

1.1 Equivalence between infinite horizon and repeated generations

We now investigate when the (non-) overlapping generations model and the infinite-horizon model are equivalent.

Assume that each individual lives for one period and has a child who lives next period. The individual has altruism towards his/her child, so preferences are given by

$$u(c_t) + \beta V(k_{t+1}),$$

where $u(c_t)$ is utility over own consumption and $V(k_{t+1})$ is the utility of the child, given an inheritance of k_{t+1} units of capital. The parameter $\beta < 1$ is the weight on child's utility (altruistic parameter). Note that since

$$\begin{aligned} V(k_{t+1}) &= u(c_{t+1}) + \beta V(k_{t+2}) \\ V(k_{t+2}) &= u(c_{t+2}) + \beta V(k_{t+3}) \\ &\ldots \end{aligned}$$

we can write

$$V(k_0) = \sum_{t=0}^{T} \beta^t u(c_t) + \beta^T V(k_T),$$

Equivalence:

Proposition 1 *if $\beta < 1$ and some weak conditions on the utility function u is satisfied (u bounded would be sufficient), then*

$$\lim_{T\to\infty} \beta^T V(k_T) = 0,$$

which implies

$$V(k_0) = \sum_{t=0}^{\infty} \beta^t u(c_t).$$

which establishes that the repeated non-overlapping generation model is equivalent to the infinite-horizon model.

Proposition 2 *Even if generations were overlapping and households are free to save over the life cycle, the equivalence result of Proposition 1 would hold.*

Comment: for all practical purposes, the difference between the behavior of the overlapping-generations model and the infinite-horizon model boils down to how strong the altruism is (i.e., the strength of the bequest motive). Reason: it is natural to assume that bequests cannot be negative.

1.2 Ricardian equivalence

- Ricardian equivalence: It is only the NPV of government expenditures that matters for households' consumption. The timing of taxes and, hence, the debt sequence are irrelevant.
- Intuition: government debt is not net wealth because government debt implies a future tax burden. When debt increases, households save so as to be able to pay the future debt
- Conditions necessary for Ricardian equivalence to hold (see Marcus' proof):
  1. Taxes are lump sum (i.e., non-distortive)
  2. The government can borrow and lend at the same terms as the private households
  3. **EITHER**
     (a) Households are infinitely-lived (with no binding borrowing constraint), **OR**
     (b) Households are finitely lived and have altruism toward their children, so their preferences are given by
     $$u(c_t) + \beta V(k_{t+1}),$$
     where $\beta \in (0,1)$ and the utility function u satisfies Proposition 1, so that the discounted utility can be written as
     $$V(k_0) = \sum_{t=0}^{\infty} \beta^t u(c_t).$$
     MOREOVER, one of the following must hold:
     i. there are no constraints on bequests (can give both negative and positive bequests), OR
     ii. the bequest motive is so strong that bequests are always positive (so a non-negativity constraint does not bind)
- These conditions give us a hint of how to break Ricardian Equivalence:

- assume that an infinitely-lived household is borrowing constrained OR assume that bequests are zero OR assume that the government can borrow and lend at rates different from the interest rates for households.
- With zero bequests (or a borrowing constrained dynasty) then a small tax break today (matched by higher taxes in the future) will increase the revenue for the current generation and, hence, increase the household's consumption.
- With non-overlapping generations the budget constraint is simply

$$c_t + k_{t+1} = (1 + r) k_t + w_t - T_t$$

where w_t is labor income, c_t is consumption, T_t is a lump-sum tax, and k_{t+1} is bequest from generation t to generation $t + 1$. Clearly, if there are no bequests (so $k_t = k_{t+1} = 0$),

$$c_t = w_t - T_t,$$

so the timing of taxes has full effect on consumption.
- With infinite horizon and a borrowing constraint (e.g., $k_{t+1} \geq 0$), the Euler equation becomes

$$u'(c_t) \geq \beta (1 + r_{t+1}) u'(c_{t+1}),$$

with equality if the borrowing constraint is not binding and inequality if it is binding (i.e., if the desired $k_{t+1} < 0$). Intuition: if the household is borrowing constrained at time t, consumption will be lower than if unconstrained, and therefore the marginal utility will be higher than if unconstrained.
  - Key implication: when more households are borrowing constrained (living had-to-mouth), the further the economy will be from Ricardian Equivalence.
  - Empirical evidence: In practice, there is a difference between small and large tax breaks:
    - LARGE propensity to consume out of small tax breaks (a clear violation of RE)
    - SMALL propensity to consume out of LARGE tax breaks, especially for the rich (consistent with RE).
  - Empirical evidence from the Bush tax cuts (SMALL) and the payouts from the Alaskan Heritage Fund (LARGE)
- *Caveat with distortive taxation*: Note that assuming that taxes are distortive will make Ricardian Equivalence not hold, but a tax cut (i.e., a delay of taxation and, hence, an increase in debt) can actually *reduce* private consumption. Consider the following two-period

example: The household problem is

$$\log c_1 + \log\left(c_2 - \frac{h_2^{1+1/\phi}}{1+1/\phi}\right)$$

subject to the budget constraint

$$c_1 + c_2 = T_1 + (1-\tau)\, h_2,$$

where h is labor supply in period 2, the pre-tax hourly wage is 1, τ is a proportional tax (distortive) and T is a lump-sum transfer in period 1. Note that there is no discounting and the interest rate is zero. Write the problem as a Lagrange problem:

$$\log c_1 + \log\left(c_2 - \frac{h_2^{1+1/\phi}}{1+1/\phi}\right) - \lambda\left[c_1 + c_2 - T_1 - (1-\tau)\, h_2\right].$$

Take first-order conditions with respect to h_1, c_1, and c_2:

$$\begin{aligned}
0 &= -\lambda + \frac{1}{c_1} \\
0 &= -\lambda + \frac{1}{c_2 - \frac{h_2^{1+1/\phi}}{1+1/\phi}} \\
0 &= -\frac{h_2^{1/\phi}}{c_2 - \frac{h_2^{1+1/\phi}}{1+1/\phi}} + \lambda\,(1-\tau)
\end{aligned}$$

Rewrite this as

$$\begin{aligned}
h_2 &= (1-\tau)^{\phi} \\
c_1 &= c_2 - \frac{(1-\tau)^{\phi+1}}{1+1/\phi}
\end{aligned}$$

Substitute this back into the budget constraint to obtain

$$\begin{aligned}
c_1 + c_1 + \frac{(1-\tau)^{\phi+1}}{1+1/\phi} &= (1-\tau)^{\phi+1} + T_1 \\
&\Rightarrow \\
c_1 &= \frac{1}{2}\frac{1}{\phi+1}(1-\tau)^{\phi+1} + \frac{T_1}{2}
\end{aligned}$$

Suppose the government issues a lumps-sum transfer T_1, financed by debt, and repay the debt next period by issuing distortive taxes. The budget constraint for the government is

$$\begin{aligned}
T_1 &= \tau h = \tau\,(1-\tau)^{\phi} \\
&\Rightarrow \\
T_1 &= \tau\,(1-\tau)^{\phi}
\end{aligned}$$

Consumption then becomes

$$c_1 = \frac{1}{2}\frac{1}{\phi+1}(1-\tau)^{\phi+1} + \frac{\tau(1-\tau)^{\phi}}{2},$$

which is decreasing in τ since

$$\frac{\partial c_1}{\partial \tau} = -\frac{1}{2}\tau\phi(1-\tau)^{\phi-1} < 0$$

* Conclusion: a lump sum transfer will *lower* current consumption because it must be financed by (future) tax revenue from distortive taxation.

1.3 An application to pension systems

- There are two types of pension systems: pay-as-you-go and fully funded systems

  1. Pay-as-you-go: no accumulation of funds. Every period benefits are paid from current taxes:

     $$\text{benefits} = T$$

     – Crowds out private savings (because households have less disposable income when young and more (pension) income when old
     – Since government savings does not change, it crowds out aggregate savings and, hence lower the capital stock. To see this, consider the market-clearing condition for the savings in a closed economy:

     $$S_t = b + k$$

     where b is government debt, k is capital, and S_t is private savings.

  2. Fully funded system: government saves the pension-tax revenue, so no effect on aggregate savings (the accumulation of the pension fund h matches reduction in private savings)

     $$S_t + h = b + k$$

     – Note: no need for such pension system unless some people are irrational (rational households can save on their own)

- All industrialized countries have mandatory pension schemes. Across countries, these systems have several features in common:

  – were put in place between 1930-1960 and expanded during 1960-1980.
  – pension contributions are, legally, a loan to the government from the worker, paying a particular return x.

- pension contributions are subtracted from earnings before the employer gets to pay the worker (a payroll tax).
- pension systems contain an old-age component and a spouse component. In some countries the pension system also provide medical insurance and finance early retirement.
- initially, the systems were all pay-as-you-go, or balanced within each period, i.e.,

$$
\begin{aligned}
0 &= N_{t-1} T_t^o + N_t T_t^y \\
&\Rightarrow \\
-T_t^o &= \frac{N_t}{N_{t-1}} = (1+n) T_t^y,
\end{aligned}
$$

  where N_t is the size of the cohort born in period t and T^y and T^o are taxes issued on the young and the old. Thus, $N_t/N_{t-1} = 1+n$ is the population growth and $1+n$ is also the *old-age dependency ratio*, i.e., number of workers per retiree)
- Due to the population transition (lower fertility after 1960 and longer longevity), most countries now promise a return and accumulate a pension fund to finance future pension liabilities for the "baby-boomers".
- The introduction of pension systems worked as a great transfer of wealth to the initial old.
- The implied rate of return on pay-as-you-go pension contributions, x_t, is, on average, the aggregate growth rate of labor earnings. In our simple economies, this return is simply

$$
1 + x_t = \frac{N_t}{N_{t-1}}
$$

  Thus, if the pension contribution for the young is a fixed fraction η of the endowment when young (i.e., a proportional pension tax η), the consumption allocations will be

$$
\begin{aligned}
c_t^y &= \omega^y - T_t^y - a_{t+1} = (1-\eta)\,\omega^y - a_{t+1} \\
c_{t+1}^o &= \omega^o - T_{t+1}^o + (1 + r_{t+1})\, a_{t+1} \\
&= \omega^o + (1 + x_{t+1})\,\eta\omega^y + (1 + r_{t+1})\, a_{t+1}
\end{aligned}
$$

  where c_t^y and c_t^o are consumption allocations for young and old (in period t), ω^y and ω^o are endowments (earnings) for young and old, a_{t+1} is savings of the young, and r_{t+1} is an interest rate on that savings.

- The present value budget constraint then becomes

$$\begin{aligned}
c_t^y + \frac{c_{t+1}^o}{1+r_{t+1}} &= (1-\eta)\,\omega^y + \frac{\omega^o + (1+x_{t+1})\,\eta\omega^y}{1+r_{t+1}} \\
&= \left[1-\eta+\frac{1+x_{t+1}}{1+r_{t+1}}\eta\right]\omega^y + \frac{\omega^o}{1+r_{t+1}} \\
&= \underbrace{\omega^y + \frac{\omega^o}{1+r_{t+1}}}_{\text{NPV of before-tax earnings}} + \underbrace{\left(\frac{1+x_{t+1}}{1+r_{t+1}}-1\right)\eta\omega^y}_{\text{Net gain on the pension system}}
\end{aligned}$$

- *Conclusion 1: A pay-as-you-go pension system is, on the margin, a gain, in terms of the present value of consumption, if and only if $x_{t+1} > r_{t+1}$ (dynamic inefficiency). Conversely, if $x_{t+1} < r_{t+1}$ (dynamic efficiency), the pension system works as a tax (i.e. mandatory savings at a below-market rate of return).*
- For simplicity, assume $\omega^o = 0$ and that the utility function is $U_t = \log(c_t^y) + \beta\log\left(c_{t+1}^o\right)$. This implies

$$\begin{aligned}
c_t^y &= \frac{1}{1+\beta}\cdot NPV\,(\text{after-tax wealth}) \\
&= \frac{1}{1+\beta}\left[1+\left(\frac{1+x_{t+1}}{1+r_{t+1}}-1\right)\eta\right]\omega^y.
\end{aligned}$$

  Aggregate private savings are then given by

$$\begin{aligned}
S_t^y &= (1-\eta)\,\omega^y - c_t^y \\
&= (1-\eta)\,\omega^y - \frac{1}{1+\beta}\left[1+\left(\frac{1+x_{t+1}}{1+r_{t+1}}-1\right)\eta\right]\omega^y \\
&= \left[\frac{\beta}{1+\beta} - \left(\frac{\beta}{1+\beta}+\frac{1}{1+\beta}\frac{1+x_{t+1}}{1+r_{t+1}}\right)\eta\right]\omega^y
\end{aligned}$$

- *Conclusion 2: the pension system crowds out private savings*
- The aggregate annual growth rate of wages has been 2-4% in most OECD countries during the last 50 years (roughly 1-2% population growth rate and roughly 1-3% growth rate in wages per worker).
- The average "riskfree" rate of return has been, on average, 1% during the 20th century (compared to 5-9% average stock market return).
- Thus, this "free lunch" may have been a major motivation for the introduction of the pension systems.
- The leading alternative motivation for the introduction of the pension systems is paternalism, the belief that policy makers know better how much individuals should save than do the individuals themselves.
- See the Diamond model for how to introduce capital in the OLG model

Is the world "dynamically inefficient"?

2 Optimal fiscal policy

- Assume the economy is growing at a constant rate γ and that the world-market interest rate is constant at r
- Law of motion for government debt is given by

$$B_{t+1} = (1+r)\,B_t + G_t - T_t$$

and in shares of GDP:

$$\begin{aligned}(1+\gamma)\,b_{t+1} &= (1+r)\,b_t + \underbrace{g_t - \tau_t}_{\text{primary deficit}} \\ &\Rightarrow \\ b_{t+1} &= \frac{1+r}{1+\gamma}b_t + \frac{g_t-\tau_t}{1+\gamma}\end{aligned}$$

- The ratio $\frac{1+r}{1+\gamma}$ determines the drift of debt – interest effect (high r increases debt burden) versus growth effect (high γ alleviates debt burden)
- Suppose the primary deficit and debt are constant over time (as a share of GDP). And suppose the debt-to-output ratio is constant ($b_{t+1} = b_t = b$). This implies

$$\begin{aligned}b &= \frac{1+r}{1+\gamma}b + \frac{g-\tau}{1+\gamma} \\ &\Rightarrow \\ b &= \frac{\tau-g}{r-\gamma}\end{aligned}$$

- Suppose, first, that $r < \gamma$ (dynamically inefficient case)
  - Could sustain for ever a deficit equal to

  $$g - \tau = (\gamma - r)\,b$$

  ... so debt will never have to be paid back (positive debt and perpetual deficits)
  - The no-Ponzi scheme condition is not binding!
- Suppose (more reasonably) that $r > \gamma$. In this case we need a perpetual primary surplus to keep debt-output ratio constant, where

$$\begin{aligned}b_t &= \text{NPV of future primary surpluses} \\ &= \sum_{j=0}^{\infty}\left(\frac{1+\gamma}{1+r}\right)^j(\tau_{t+j}-g_{t+j})\end{aligned}$$

  - so a no-Ponzi scheme condition must hold

- Note: if you want to run perpetual deficits, it is necessary with $b < 0$

- Consider some examples of the relationship between b, $r-\gamma$, and $\tau - g$. Compute the primary surplus required by a particular b and $\gamma - r$. Magnitudes are large:

Required surplus, $\tau - g$

b	$r-\gamma = 0.5\%$	$r-\gamma = 4\%$
50%	0.25%	2%
200%	1%	8%

- Compute the maximum debt that could possibly be sustained. Assume:
  - taxes are at top of Laffer curve, say $\tau = 50\%$ of GDP
  - government spending is a minimum to run a state (e.g., zero transfers and only basic services), say $g = 10\%$ of GDP

$$\bar{b} = \frac{\tau - g}{r-\gamma} = \frac{0.50 - 0.10}{r-\gamma},$$

  which implies

	$r-\gamma = 0.5\%$	$r-\gamma 4\%$
$\bar{b}$	80	10

  - Note: sovereign debt issues would kick in *long* before reaching these levels

2.1 Simple fiscal rules

Consider now a country that has a large wealth $b < 0$ (due to finding oil, say). How should the oil wealth be distributed across generations? Consider two simple rules:

1. Rule 1: all generations get the same contribution from the fund (in levels, i.e., kroner). Clearly, to keep B constant it is necessary to take out

$$-r * B$$

   every period. With for example $r = 4\%$, this gives the rule "eat 4% of fund every period"

   - Note that with positive growth ($\gamma > 0$), $B_t/Y_t \to 0$ in the long run

2. Rule 2: all generations get a take-out from the fund equal to the same share of their GDP

   - Motivation: government services might be produced using workers for which there is little productivity growth (e.g. teachers or the military)

- Necessary to keep $b_t = B_t/Y_t$ constant
- The take out (i.e., long-run primary deficit as a share of GDP) is then given by
$$g - \tau = -(r - \gamma)\, b$$
With e.g. $r - \gamma = 2\%$ and $-b = 4$ (optimistic view of the Norwegian case), we get
$$g - \tau = 2\% \cdot 4 = 8\%.$$
As a share of the value of the fund this becomes
$$\frac{g - \tau}{b} = r - \gamma = 2\%,$$
i.e., only half the current rate of extraction.

2.2 Solve for the optimal rule

- Return to the model with non-overlapping generations. To simplify the exposition, we abstract from population growth. Assume that people have little concern for their children so bequests are zero in equilibrium. Consider the problem of a planner who weights the utility of generation t with a discount factor β^t, where $\beta < 1$. The objective function of the planner is therefore
$$\max \sum \beta^t u(c_t)$$
- Small open economy, and the planner faces a constant interest rate r and there is no default risk (this interest rate is not affected by the planner's actions, given the assumption of a small open economy and no default risk).
- Generation t has labor earnings w_t, which grows at rate g (assumed to be $g < r$),
$$w_t = (1 + g)^t\, w$$
- The planner can freely decide consumption across different households, subject to a no-Ponzi scheme condition,
$$\begin{aligned} A_0 &= \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} (c_t - w_t) \\ &= \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} \left(c_t - w\,(1+g)^t\right), \end{aligned}$$
where A_0 is the planner's initial assets.
- Question 1: what is the optimal path of consumption?

- Answer: the Euler equation gives

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta(1+r)$$

- Assume that preferences are of the constant relative risk aversion class, i.e.,

$$u(c) = \frac{c^{1-\gamma} - 1}{1-\gamma}$$

where $1/\gamma$ is the intertemporal elasticity of substitution

- The Euler equation then becomes

$$\frac{c_{t+1}}{c_t} = [\beta(1+r)]^{\frac{1}{\gamma}}$$

so the planner's optimal consumption stream is

$$c_t = (\beta(1+r))^{t/\gamma} c_0$$

- Question 2: Assume that the government finds oil so A_0 becomes large. How should the fund be distributed over time?

- To answer questions 1 and 2 we must take a stand on the planner's discount factor

2.2.1 Nordhaus' discount rate

- Suppose the discount rate is the interest rate (Nordhaus), i.e., $\beta = 1/(1+r)$.

- The the planner's optimal consumption stream is then

$$c_t = \left(\frac{1}{1+r}(1+r)\right)^{t/\gamma} c_0 = c_0$$

... so optimal consumption is constant over time, irrespective of the utility function and the wage growth

- The resource constraint then becomes

$$\begin{aligned}
A_0 &= \sum_{t=0}^{\infty} \frac{1}{(1+r)^t}\left(c_0 - w(1+g)^t\right) \\
&\Rightarrow \\
\frac{A_0}{1+r} &= \frac{c_0}{r} - \frac{w}{r-g} \\
c_0 &= c_t = r \cdot \left(\frac{A_0}{1+r} + \frac{w}{r-g}\right),
\end{aligned}$$

i.e., a constant fraction r out of total wealth (financial assets $A_0/(1+r)$ plus the present value of future labor income)

- Initially, wages are low but consumption relatively high, so next-period financial wealth is falling:

$$\begin{aligned} A_1 &= (1+r)(A_0 + w_0 - c_0) \\ &= (1+r)\left[A_0 + w - r \cdot \left(\frac{A_0}{1+r} + \frac{w}{r-g}\right)\right] \\ &= A_0 - (1+r)\, w \frac{g}{r-g} < A_0 \end{aligned}$$

- Conclusion: Since wages are increasing over time, the government must throw big party initially and huge debt build-up later ("enslave" future generations)
- Implication: if there is an initial oil fund ($A_0 > 0$), the fund will be run down trust fund fast

2.2.2 Stern's discount rate

- Suppose the discount rate is the interest rate minus the growth rate times γ (Stern), i.e., $\beta = (1+g)^{\gamma} / (1+r)$ so the discount *rate* is approximately $r - \gamma g$.
- Natural benchmark because it implies no redistribution in steady state (unless the government has some wealth)
- Implies that optimal consumption is growing at rate g over time at the same rate as wages:

$$c_t = \left(\frac{(1+g)^{\gamma}}{(1+r)}(1+r)\right)^{t/\gamma} c_1 = (1+g)^t c_0$$

- If initial government assets are zero, the optimal allocation is autarky: $c_t = w_t$ and keeping $A_t = 0$.
- The resource constraint becomes

$$\begin{aligned} A_0 &= \sum_{t=0}^{\infty} \frac{1}{(1+r)^t}\left((1+g)^t c_0 - w(1+g)^t\right) \\ &\Rightarrow \\ c_0 &= (r-g) \cdot \frac{A_0}{1+r} + w \\ c_t &= (r-g) \cdot \frac{A_0}{1+r}(1+g)^t + w_t \\ &= \left[(r-g) \cdot \frac{A_0}{1+r} + w\right](1+g)^t \end{aligned}$$

- The trust fund must then be maintained at a constant size relative to wages

- Since wages are increasing over time at rate g, the trust fund must increase over time at rate g
- Take-out from fund must then be $T_t = (r - g) A_t$

2.2.3 Handlingsregelen

- Return to Handlingsregelen (i.e., a constant take-out rA_t)
- Implies a constant wealth $A_t = A_1$ (assuming the fund can deliver a return r)
- Over time, trust fund will become irrelevant relative to wages:

$$\frac{A_t}{(1+g)^t w} = \frac{1}{(1+g)^t} \frac{A_0}{w} \to 0$$

- Discussion:
  - What is the implied inter-generational discount rate? Answer: higher than $r - g$ in the short run (the first 100 years, say) and equal to $r - g$ in the long run
  - Motivation (?): future generations are much richer, so it is fair that the current ones get more as a share of their GDP. Implies a discount rate closer to Nordhaus in the short run and close to Stern in the long run
  - Rule was agreed upon in 2000. at that time, the long-run real interest rates were 3-4% and with an even higher return to capital (due to a risk premium, say), it seemed conservative to go for a 4% rule
  - Current long (30-year) interest rates on debt are low (and have fallen a lot, from 3% to about 1%). Assuming an unchanged risk premium, a rule preserving the size of B should be lower (2-3%, perhaps) and a rule holding B/Y constant must be around 1-2%.

2.3 Risk and portfolio choice

2.3.1 Ricardian equivalence of portfolio choice

- Consider a world with two assets – capital k and (foreign) bonds b. Suppose the return on capital is risky (with return $\tilde{r}_t$) and that the bonds are risk free (with return r). Maintain the small open economy assumption, i.e., domestic wages are no affected by the portfolio choice of households and/or government. Abstract from domestic wage growth (so $w_t = w_0$)
- Let the portfolio for the government be (b^g, k^g). Budget constraint for the government is

$$b^g_{t+1} + k^g_{t+1} = (1+r) b^g_t + (1+\tilde{r}_t) k^g_t + \tau_t - g_t$$

- Households: Suppose households are infinitely lived. Let the portfolio for the household be (b^p, k^p). If the financial markets are perfect (no transaction costs, no portfolio restrictions, and same return for all investors), the budget constraint for the households is

$$b_{t+1}^p + k_{t+1}^p = (1+r)\, b_t^p + (1+\tilde{r}_t)\, k_t^p + w_t - \tau_t$$

- An extended version of Ricardian equivalence:

1. Timing of taxes is irrelevant for the households' consumption allocations
2. Government's portfolio choice is irrelevant for the households' consumption allocations

- - Conclusion: neither government saving nor the government fund's portfolio matter for individual consumption
  - Intuition: households will undo the governments' actions if they disagree

- In what cases would the oil fund's portfolio allocations matter?
  - If the government fund delivers excess returns (i.e., beats the market return available to private households, $\tilde{r}_t^g > \tilde{r}_t$ or $r_t^g > r_t$, without taking higher risk), one can sustain higher g and/or lower τ
  - If Ricardian equivalence does not hold. For example, if private households are portfolio constrained (i.e., they cannot purchase the optimal portfolio) and/or there are no private bequests between generations, then the fund is effectively doing the dynastic portfolio allocation on behalf of the households

2.3.2 Portfolio choice: a simple example

What portfolio should the country choose? Consider the following two-period economy

- There are two countries $i \in \{A, B\}$ of equal size, each with a representative household with utility $u(c_i)$ in period 1 (where u satisfies the standard properties: concave and monotone increasing).
- In period zero each household in country i own a tree which will yield fruits y_i in period 1
- The fruits are risky;

$$\begin{aligned} y_A &= y + \varepsilon \\ y_B &= y - \varepsilon, \end{aligned}$$

  where $\varepsilon = x$ with 50% probability and $\varepsilon = -x$ with 50% probability. Moreover, $y > 0$ is also a stochastic variable.

- Here y is an aggregate shock (to the world economy) and ε is a country-specific "idiosyncratic" shock.
- Claims to the trees can be traded freely at prices p_A and p_B for tree A and B, respectively.
- The budget constraint for household A (the "home country") is

$$1 \cdot p_A = z_A \cdot p_A + z_B \cdot p_B, \tag{1}$$

  where z_A and z_B are the household's holdings of tree A and B, respectively, after trading. Similarly, z_A^* and z_B^* are the "foreign" household's (i.e., households in country B) holdings of tree A and B.
- The home country household's problem is to solve

$$\begin{aligned} & \max_{z_A, z_B} u(c) \\ & \text{subject to} \\ 1 \cdot p_A &= z_A \cdot p_A + z_B \cdot p_B \\ c_A &= z_A \cdot (y+\varepsilon) + z_B \cdot (y-\varepsilon) \end{aligned}$$

- **Definition of a competitive equilibrium**: A competitive equilibrium is a consumption allocation $\{c_A, c_B\}$, a price vector $\{p_A, p_B\}$, and a portfolio allocation $\{z_A, z_B, z_A^*, z_B^*\}$ so that all markets clear and the portfolio allocation solves the optimization problems for each household. Define tree A as the numeraire, so $p_A = 1$.
- Market clearing requires that supply equals demand for both trees and for aggregate consumption, i.e.,

$$\begin{aligned} 1 &= z_A + z_A^* \\ 1 &= z_B + z_B^* \\ y_A + y_B &= c + c^* \end{aligned}$$

- Guess that $p_B = p_A = 1$ and $z_A = z_B = z_A^* = z_B^* = 1/2$, which implies $c = c^* = y$
- What do we learn from this example?
  1. All households share equally the aggregate shock (c and c^* are equally exposed to the aggregate shock)
  2. Households insure perfectly the idiosyncratic shock (ε does not affect consumption c and c^*)

2.3.3 General messages for portfolio holdings:

- Claim 1: **Market Portfolio**. If the country has the same endowments as the other world investors: hold the market portfolio. Proof: "macro consistency". Idea: if it is optimal to deviate from the rest of the world investors, then the others must be making mistakes. Moreover, the only way all investors can do the same is if they all hold equal shares in all stocks and bonds.

- Claim 2: **Risk Sharing.** Suppose households have the same CRRA preferences as the rest of the world. Then the portfolio should be chosen so as to share risk with the world investor, i.e.,

$$\frac{c_{t+1}}{c_t} = \frac{C_{t+1}^w}{C_t^w}, \tag{2}$$

  where c_t is domestic consumption in period t and C_t^w is world consumption in period t.

- Claim 3: **Exploit Differences.** If the country is different from the world: portfolio should be adjusted so as to achieve risk sharing with the rest of the world (2)
  - Example 1: if country has future oil endowments (and will sell oil to the rest of the world) and future oil price is risky, optimal to share risk by locking in future oil price
  - Example 2: The "risk premium" (compensation for holding stocks) offered by world's financial markets is much higher in recessions than in normal times. If domestic government's risk tolerance is constant, then one should purchase stocks in recession (when stocks fall in value) and sell in booms. This is labeled "rebalancing rule"