1 Overlapping Generations

1.1 Motivation

- So far: infinitely-lived consumer. Now, assume that people live finite lives.
- Purpose of lecture:
  - Analyze a model which is of interest in its own right (and which can give quite different implications than the infinite-horizon model)
  - Break the 1st welfare theorem (i.e., that c.e. is Pareto efficient)
  - Break Ricardian equivalence and the implication that $k^* < k^g$.
  - Real world: Study rational bubbles and pension schemes

1.2 The environment

- Preferences
  - People live for two periods, young and old
  - They care about consumption when young c_t^y and consumption when old c_{t+1}^o. For simplicity, assume additive separable preferences:

$$u\left(c_t^y, c_{t+1}^o\right) = u\left(c_t^y\right) + u\left(c_{t+1}^o\right)$$

    When writing down a specific utility function, we will use $u(c) = \log c$
  - Note: no altruism (with altruism and bequests, the dynamics of the model would be as in the infinite-horizon model)
  - For simplicity, assume zero population growth (N young and N old individuals).
- Technology
  - Abstract from production (will do it next lecture)
  - Assume that people have endowments ω_t^y when young and ω_{t+1}^o when old of the consumption good. Interpretation: fruit that falls down next to bed, or time endowment for picking blueberries
  - For simplicity:
    - Focus on *stationary* endowments, where

$$\begin{aligned} \omega_t^y &= \omega^y \\ \omega_t^o &= \omega^o \end{aligned}$$

      for all time periods $t \geq 1$.
    - All individuals in a generation have the same endowment

- Start with no government (will introduce later)
- Markets:
  - There is a one-period bond (in zero net supply). Purchase one unit in period t. Pay back, with interest, $1 + r_{t+1}$ in period $t + 1$.
  - Key imperfection: cannot trade with the unborn.
  - Note: there are no long-lived assets
- The budget constraints for the individuals are:
  - for all individuals born in period $t = 1$ or later:

$$\begin{aligned} c_t^y + b_{t+1} &= \omega^y \\ c_{t+1}^o &= (1 + r_{t+1})\, b_{t+1} + \omega^o \end{aligned}$$

  - For now, assume that the initially old have no initial assets. Since they cannot pay back in period $t = 2$, their budget constraint must simply be $c_1^o = \omega^o$.

1.3 Solving for the equilibrium without a government

Definition 1 *A competitive equilibrium is defined as an allocation* $\{c_t^y, c_t^o, b_{t+1}\}_{t=1}^{\infty}$ *and a price sequence* $\{r_t\}_{t=2}^{\infty}$ *such that*

1. The consumption allocation $\{c_t^y, c_t^o\}_{t=1}^{\infty}$ solves the optimization problem for every generation born in period t and later, where households take the price sequence $\{r_t\}_{t=2}^{\infty}$ as given:

$$\begin{aligned} & \max \left\{ u\left(c_t^y\right) + u\left(c_{t+1}^o\right) \right\} \\ & \text{subject to} \\ c_t^y + b_{t+1} &= \omega^y \\ c_{t+1}^o &= (1 + r_{t+1})\, b_{t+1} + \omega^o \end{aligned}$$

   and c_1^o solves the problem for the (initial) old in period 1:

$$\begin{aligned} & \max \left\{ u\left(c_1^o\right) \right\} \\ & \text{subject to} \\ c_1^o &= \omega^o \end{aligned}$$

2. All markets clear.
   - – Bonds: the net demand for bonds is zero in every period $t \geq 1$:

$$0 = \sum_{i=1}^{N} b_{t+1}^i$$

– Goods:

$$\begin{aligned}
\sum_{i=1}^{N} c_t^{y,i} + \sum_{i=1}^{N} c_t^{o,i} &= \sum_{i=1}^{N} \omega^y + \sum_{i=1}^{N} \omega^o \\
N c_t^y + N c_t^o &= N\omega^y + N\omega^o
\end{aligned}$$

- Discussion:
  - Since all households are identical, they all demand the same number of bonds ($b_{t+1}^i = b_{t+1}$), so the condition $0 = \sum b_{t+1}^i$ is equivalent to $b_{t+1} = 0$.
  - There are no possibilities for the old to pay back to or get paid by the young next period (because they are dead). Therefore, there cannot be any trade between generations

- Solution:
  - Solve the individual optimization problem (substitute out c^y and c^o):

$$\begin{aligned}
& \max_{b_{t+1}} \left\{ u\left(\omega^y + b_{t+1}\right) + u\left(\left(1 + r_{t+1}\right) b_{t+1} + \omega^o\right) \right\} \\
& \Rightarrow \\
0 &= -u'\left(\omega^y - b_{t+1}\right) + \left(1 + r_{t+1}\right) u\left(\left(1 + r_{t+1}\right) b_{t+1} + \omega^o\right) \\
\frac{u'\left(\omega^y - b_{t+1}\right)}{u\left(\left(1 + r_{t+1}\right) b_{t+1} + \omega^o\right)} &= \frac{u'\left(c_t^y\right)}{u\left(c_{t+1}^o\right)} = \left(1 + r_{t+1}\right),
\end{aligned}$$

  i.e., the Euler equation with $\beta = 1$ (there is no restriction on β in an OLG economy).
  - Since $b_{t+1} = 0$ for all $t \geq 1$, the competitive equilibrium allocation must be, for all t,

$$\begin{aligned}
c_t^y &= \omega^y \\
c_t^o &= \omega^o.
\end{aligned}$$

  - Derive the prices

$$\begin{aligned}
\frac{u'\left(\omega^y\right)}{u'\left(\omega^o\right)} &= \left(1 + r_{t+1}\right) \\
& \Rightarrow \\
\left(1 + r_{t+1}\right) &= \left(1 + r\right) = \frac{u'\left(\omega^y\right)}{u'\left(\omega^o\right)}
\end{aligned}$$

  - If we set $u\left(c\right) = \log c$, we get

$$\left(1 + r\right) = \frac{\omega^o}{\omega^y}$$

  - Note that the interest rate can be both positive and negative, depending on whether $\omega^o >< \omega^y$

1.4 Dynamic inefficiency

- Consider a case when $\omega^o < \omega^y$ and $r < 0$. Propose a feasible reallocation:

  1. Every period, young give
  $$\Delta = \omega^y - \frac{\omega^y + \omega^o}{2} = \frac{\omega^y - \omega^o}{2}$$
  to the old
  2. Thus, new allocation is
  $$c_t^y = c_t^o = \frac{\omega^y + \omega^o}{2}$$

- Claim: all generations are *better off*
  - The initially old get to consume $\frac{\omega^y+\omega^o}{2} > \omega^o$ and are better off
  - All newborn get utility
  $$u\left(\frac{\omega^y + \omega^o}{2}\right) + u\left(\frac{\omega^y + \omega^o}{2}\right) > u(\omega^y) + u(\omega^o)$$

- Note that any transfer $\Delta \leq (\omega^y - \omega^o)/2$, i.e.,
  $$\Delta \in \left[0, \frac{\omega^y - \omega^o}{2}\right],$$
  would be a Pareto improvement (even larger values for Δ would be an improvement)

- This is an example of *dynamic inefficiency.* Have dynamic inefficency whenever $r < 0$ (same condition as in the Solow model: $f'(k^*) < \delta+n+g$)

- Note that the competitive equilibrium is inefficient, so the first welfare theorem breaks down

- Reason: many missing markets (the unborn cannot trade). There is a "shortage of assets"

1.5 Introducing a government (but not yet debt)

- A government is viewed as an infinitely lived and time consistent institution.

- The government can issue lump-sum taxes on the yong and the old, T_t^o and T_t^y. Note that negative taxes (e.g., $T_t^o < 0$) is the same as transfers

- Since there is no government debt, their budget constraint is, for all t,

$$0 = T_t^y + T_t^o$$

Definition 2 *A competitive equilibrium is defined as an allocation* $\{c_t^y, c_t^o, b_{t+1}, T_t^o, T_t^y\}_{t=1}^{\infty}$ *and a price sequence* $\{r_t\}_{t=2}^{\infty}$ *such that*

1. The consumption allocation $\{c_t^y, c_t^o\}_{t=1}^{\infty}$ solves the optimization problem for every generation born in period t and later, where households take the price sequence $\{r_t\}_{t=2}^{\infty}$ and fiscal policy $\{b_{t+1}, T_t^o, T_t^y\}_{t=1}^{\infty}$ as given:

$$\begin{aligned} & \max\left\{u\left(c_t^y\right) + u\left(c_{t+1}^o\right)\right\} \\ & \text{subject to} \\ c_t^y + b_{t+1} &= \omega^y - T_t^y \\ c_{t+1}^o &= \left(1 + r_{t+1}\right) b_{t+1} + \omega^o - T_t^o \end{aligned}$$

and c_1^o solves the problem for the (initial) old in period 1:

$$\begin{aligned} & \max\left\{u\left(c_1^o\right)\right\} \\ & \text{subject to} \\ c_1^o &= \omega^o - T_1^o \end{aligned}$$

2. All markets clear. Namely, the net demand for bonds is zero in every period $t \geq 1$:

   - – Bonds: the net demand for bonds is zero in every period $t \geq 1$:

$$0 = \sum_{i=1}^{N} b_{t+1}^i = b_{t+1}$$

     – Goods:

$$c_t^y + c_t^o = \omega^y + \omega^o$$

     – Government's budget constraint holds:

$$0 = T_t^y + T_t^o$$

- Solution:

  – Solve the individual optimization problem (substitute out c^y and c^o). As above, the solution is given by the Euler equation:

$$\left(1 + r_{t+1}\right) = \frac{u'\left(c_t^y\right)}{u\left(c_{t+1}^o\right)} = \frac{u'\left(\omega^y - b_{t+1} - T_t^y\right)}{u\left(\left(1 + r_{t+1}\right) b_{t+1} + \omega^o - T_{t+1}^o\right)}$$

  – Focus on stationary transfer policy, i.e., $T_t^y = T^y \equiv T$. The government budget constraint then implies $T^o = -T$. Interpret $T > 0$ as a *pay-as-you-go* pension system.

– Since $b_{t+1} = 0$ for all $t \geq 1$, the competitive equilibrium allocation must be, for all t,

$$\begin{aligned} c_t^y &= \omega^y - T \\ c_t^o &= \omega^o + T. \end{aligned}$$

– Derive the prices

$$\begin{aligned} \frac{u'(\omega^y)}{u'(\omega^o)} &= (1 + r_{t+1}) \\ &\Rightarrow \\ (1 + r_{t+1}) &= (1 + r) = \frac{u'(\omega^y - T)}{u'(\omega^o + T)} \end{aligned}$$

– If we set $u(c) = \log c$, we get

$$(1 + r) = \frac{\omega^o + T}{\omega^y - T}$$

– Note that $r \uparrow$ as $T \uparrow$

- Conclusion: introducing a pension system can be Pareto improving if the economy is dynamically inefficient. Any transfer

$$0 \leq T \leq (\omega^y - \omega^o)/2$$

would be a Pareto improvement.

1.6 Introducing government debt

- Assume the government issues one-period bonds; claims to one unit of the consumption good next period. Moreover, the government always honors its debt (as before, only the young are interested in purchasing bonds). Therefore, the return on debt must be the return on private lending, r_{t+1}. If the price of one-period debt is q_t in period t, q_t must be given by

$$\begin{aligned} 1 &= q_t(1 + r_{t+1}) \\ \frac{1}{q_t} &= 1 + r_{t+1} \end{aligned}$$

- Suppose the government issues b_t units of bonds in period t. There are four ways the government can finance repayment of the debt in period $t + 1$:

1. tax the young of generation $t + 1$ a total of $T_{t+1}^y = b_t$ units
2. tax the old of generation t a total of $T_{t+1}^o = b_t$ units
3. issue b_{t+1} units of bonds that raise a total of b_t units

4. some mix of 1-3.

- The government budget constraint is

$$q_t b_t = b_{t-1} - T_t^y - T_t^o$$

  - Constraint on government: someone must be willing to buy the debt

- Budget constraint of the old (who hold b_{t-1} units of bonds):

$$c_t^o = \omega^o - T_t^o + b_{t-1}.$$

- Budget constraint of the young:

$$c_t^y = \omega^y - T_t^y - q_t b_t.$$

  Therefore, the net demand for bonds (i.e., aggregate savings) is equal to

$$q_t b_t = \omega^y - T_t^y - c_t^y \equiv S_t^y.$$

- Equilibrium definition is the same as above, except for the market for bonds

  - Supply of bonds (i.e., finance need) is given by

$$q_t b_t = b_{t-1} - T_t^y - T_t^o$$

  - Demand for bonds is given by (recall that only the young buy bonds)

$$S_t^y = \omega^y - T_t^y - c_t^y.$$

  - Hence, market clearing in the bond market now requires that

$$\begin{aligned} \omega^y - T_t^y - c_t^y &= S_t^y = b_{t-1} - T_t^y - T_t^o \\ &\Rightarrow \\ c_t^y &= \omega^y + T_t^o - b_{t-1} \end{aligned}$$

    or, equivalently, that $S_t^y = q_t b_t$

  - Note that when imposing the government budget constraint and the individual budget constraint, the market for goods clears,

$$c_t^y + c_t^o = \omega^y + \omega^o.$$

- Solve for the equilibrium. Use three equilibrium conditions: individual optimization for the young (Euler equation), optimization for the old (they consume their wealth), and the bond-market clearing equation:

$$\begin{aligned} \frac{1}{q_t} &= 1 + r_{t+1} = \frac{u'\left(c_t^y\right)}{u\left(c_{t+1}^o\right)} \\ c_t^o &= \omega^o - T_t^o + b_{t-1} \\ c_t^y &= \omega^y - T_t^y - b_{t-1} \end{aligned}$$

– Note that the debt b_{t-1} and the transfers to the old, T_t^o, cannot be too large, since $c_t^y \geq 0$ (a similar constraint for the old imposes further restrictions on b_{t-1}).

- Rolling over the debt: Suppose the government tries to just roll over the debt (i.e., set all future taxes to zero, $T_t^y = T_t^o = 0$ for all future t). What would happen?

  – The law of motion for debt would be

$$\begin{aligned} q_t b_t &= b_{t-1} \\ &= \frac{b_t}{1+r_{t+1}} \\ &\Rightarrow \\ b_t &= (1+r_{t+1})\, b_{t-1} \end{aligned}$$

  – Equilibrium conditions are

$$\begin{aligned} S_t^y &= q_t b_t = \frac{b_t}{1+r_{t+1}} \\ S_{t+1}^y &= q_{t+1} \cdot b_{t+1} = \frac{1}{1+r_{t+2}} \cdot (1+r_{t+2})\, b_t = b_t = (1+r_{t+1})\, b_{t-1} \\ S_{t+2}^y &= q_{t+2} \cdot b_{t+2} = \frac{1}{1+r_{t+3}} \cdot (1+r_{t+3})\, b_{t+1} = b_{t+1} = (1+r_{t+2})\,(1+r_{t+1})\, b_{t-1} \\ &\ldots \\ S_{t+j}^y &= (1+r_{t+j}) \cdot \ldots \cdot (1+r_{t+2})\,(1+r_{t+1})\, b_{t-1} = b_{t-1} \cdot \prod_{k=1}^{j} (1+r_{t+k}) \end{aligned}$$

  – Consider three different cases:

    1. Case 1: $r_{t+k} = 0$ for all k. Then

$$b_{t+j} = b_t$$

       and the amount of debt is constant over time.

    2. Case 2: $r_{t+k} \leq \bar{r} < 0$ for all k (and also $r_{t+k} > -1$, of course). Then

$$b_{t+j} = b_{t-1} \cdot \prod_{k=1}^{j} (1+r_{t+k}) \leq b_{t-1} \cdot \prod_{k=1}^{j} (1+\bar{r}) = b_{t-1} \cdot (1+\bar{r})^j$$

       As time moves on we have

$$0 \leq \lim_{j\to\infty} b_{t+j} \leq b_{t-1} \cdot \lim_{j\to\infty} (1+\bar{r})^j = 0,$$

       so government debt goes to zero in the long run (another stationary equilibrium, different from the one in Case 1).

3. Case 3: $r_{t+k} \geq \bar{r} > 0$ for all k. Then

$$b_{t+j} = b_{t-1} \cdot \prod_{k=1}^{j} (1 + r_{t+k}) \geq b_{t-1} \cdot \prod_{k=1}^{j} (1 + \bar{r}) = b_{t-1} \cdot (1 + \bar{r})^j$$

As time moves on we have

$$\lim_{j \to \infty} b_{t+j} \geq b_{t-1} \cdot \lim_{j \to \infty} (1 + \bar{r})^j = \infty,$$

so government debt goes to infinity, which cannot be an equilibrium since, eventually, the required refinancing will exceed the aggregate endowment of the young.

- Conclusion: debt can be rolled over for ever if and only if $r_{t+k} \leq 0$ for ever.
- Equivalence result:
An equilibrium with bonds can be duplicated (in terms of consumption allocations and prices) with a tax-transfer scheme balancing the budget of the government at all dates and having no government borrowing at any date.
- In our economy, suppose $\omega^y > \omega^o$ so the competitive *lassie-faire* competitive equilibrium is dynamically efficient and $(1 + r) = \omega^o / \omega^y < 1$. Consider the following candidate competitive equilibrium:
  - Assume that the interest rate is $r = 0$ (so $q = 1$)
  - set, in the first period,

    $$b_1 = (\omega^y - \omega^o) / 2,$$

    and transfer the funds to the old
  - roll over this debt for ever and never tax anybody, i.e.,

    $$\begin{aligned} b_t &= (\omega^y - \omega^o) / 2 \\ T_t^o &= T_t^y = 0 \end{aligned}$$

  - The implied consumption allocations are

    $$\begin{aligned} c_t^o &= \omega^o - T_t^o + b_{t-1} = \omega^o + \frac{(\omega^y - \omega^o)}{2} = \frac{\omega^y + \omega^o}{2} \\ c_t^y &= \omega^y - T_t^y - q_t b_t = \omega^y - 1 \cdot \frac{(\omega^y - \omega^o)}{2} = \frac{\omega^y + \omega^o}{2}. \end{aligned}$$

  - Verify that individual optimization holds at the equilibrium price $r = 0$:

    $$1 + r_{t+1} = \frac{u'\left(c_t^y\right)}{u\left(c_{t+1}^o\right)} = \frac{u'\left(\frac{\omega^y + \omega^o}{2}\right)}{u\left(\frac{\omega^y + \omega^o}{2}\right)} = 1,$$

    so this allocation is optimal.

– Verify that the market for savings clears. That is, at the interest rate $r = 0$, the young households are happy to save exactly enough to ensure that

$$S^y = 1 \cdot \frac{\omega^y - \omega^o}{2}$$

– Verifying that the government budget constraint holds is trivial:

$$\begin{aligned} q_1 b_1 &= b_0 - T_1^y - T_1^o \\ &\Rightarrow \\ 1 \cdot b_1 &= 0 - 0 - \left(- \frac{(\omega^y - \omega^o)}{2} \right) = \frac{(\omega^y - \omega^o)}{2} \\ b_{t+1} &= b_t. \end{aligned}$$

- Conclusions:
  1. The proposed allocation and $r = 0$ is a competitive equilibrium
  2. The competitive equilibrium is identical to the tax-and-transfer economy.
  3. This is an example of a break-down of Ricardian equivalence. Ricardian equivalence breaks down also when the economy is not dynamically inefficient. Government debt has the flavor of a pension scheme.
- Question to think about: What is "true" government debt? Should it include future pension payments?

1.7 An application to pension systems

- All industrialized countries have mandatory pension schemes. Across countries, these systems have several features in common:
  – were put in place between 1930-1960 and expanded during 1960-1980.
  – pension contributions are, legally, a loan to the government from the worker, paying a particular return h.
  – pension contributions are subtracted from earnings before the employer gets to pay the worker (a payroll tax).
  – pension systems contain an old-age component and a spouse component. In some countries the pension system also provide medical insurance and finance early retirement.
  – initially, the systems were all pay-as-you-go, or balanced within each period, i.e.,

$$\begin{aligned} 0 &= N_{t-1} T_t^o + N_t T_t^y \\ &\Rightarrow \\ -T_t^o &= \frac{N_t}{N_{t-1}} = (1+n)\, T_t^y, \end{aligned}$$

where N_t is the size of the cohort born in period t. Thus, $N_t/N_{t-1} = 1+n$ is the population growth and $1+n$ is also the *old-age dependency ratio*, i.e., number of workers per retiree)

- Due to the population transition (lower fertility after 1960 and longer longevity), most countries now promise a return and accumulate a pension fund to finance future pension liabilities for the "baby-boomers".
- The introduction of pension systems worked as a great transfer of wealth to the initial old.
- The implied rate of return on pay-as-you-go pension contributions, h_t, is, on average, the aggregate growth rate of labor earnings. In our simple economies, this return is simply

$$1 + h_t = \frac{N_t}{N_{t-1}}$$

Thus, if the pension contributions for the young are a fraction η of the endowment (when young), the consumption allocations will be

$$\begin{aligned} c_t^y &= \omega^y - T_t^y - a_{t+1} = (1-\eta)\,\omega^y - a_{t+1} \\ c_{t+1}^o &= \omega^o - T_{t+1}^o + (1 + r_{t+1})\,a_{t+1} \\ &= \omega^o + (1 + h_{t+1})\,\eta\omega^y + (1 + r_{t+1})\,a_{t+1} \end{aligned}$$

The present value budget constraint then becomes

$$\begin{aligned} c_t^y + \frac{c_{t+1}^o}{1 + r_{t+1}} &= (1-\eta)\,\omega^y + \frac{\omega^o + (1 + h_{t+1})\,\eta\omega^y}{1 + r_{t+1}} \\ &= \left[1 + \left(\frac{1 + h_{t+1}}{1 + r_{t+1}} - 1\right)\eta\right]\omega^y + \frac{\omega^o}{1 + r_{t+1}} \end{aligned}$$

- Conclusion: A pay-as-you-go pension system is, on the margin, a gain, in terms of the present value of consumption, if $h_{t+1} > r_{t+1}$. Conversely, if $h_{t+1} < r_{t+1}$, the pension system works as a tax (i.e. mandatory savings at a below-market rate of return).
- The aggregate annual growth rate of wages has been $2-4\%$ in most OECD countries during the last 50 years. Population growth rate and growth rate in wages per worker $1-2\%$).
- The average "riskfree" rate of return has been, on average, 1% during the 20th century (compared to 5-9% average stock market return).
- This "free lunch" may have been a major motivation for the introduction of the pension systems. The leading alternative motivation for the introduction of the pension systems is paternalism, the belief that policy makers know better how much individuals should save than do the individuals themselves.
- We will return to this argument when introducing capital in the OLG model

2 Bubbles and more on pensions

Purpose of lecture:

1. Study rational bubbles
2. More on pensions and fiscal rules

2.1 Long-lived assets in the OLG model

- Suppose there exists A units of a long-lived asset in the OLG economy (”land”, say). The asset pays a (constant) dividend $d_t = d$ every period.
- Let $p_{t+1}^{e,i}$ be the expectation of houshold i about the price per unit of the asset next period
  - Claim: all households will have the same expectations (assuming there are no frictions and no limits to betting),

    $$p_{t+1}^{e,i} = p_{t+1}^{e}$$

    Proof: if people held different expectations, they would bet against each other so as to align the expectations
  - Comment: the assumption about unlimited and frictionless betting is clearly violated in some markets, for example housing market: it is difficult to go short – i.e., have negative housing – and it is expensive to hold more than one house (due to moral hazard when renting out).
- Consider the payoff from purchasing the asset today and selling it tomorrow, after collecting the dividend.
  - Cost of investment is p_t
  - The (discounted) expected return on the investment is

    $$\frac{p_{t+1}^{e} + d_{t+1}}{1 + r_{t+1}}$$

  - Any equilibrium must have the expected return on the asset equal to the rate of return on private lending/bonds (otherwise there would be an arbitrage opportunity: borrow in the low-return asset and invest in the high-return asset):

    $$1 + r_{t+1} = \frac{p_{t+1}^{e} + d_{t+1}}{p_t}$$

  - This gives us a new *equilibrium condition* for the price of the asset
- Perfect foresight

- *Definition 1*: a *temporary equilibrium* is a competitive equilibrium in period t, given an expected price p^e_{t+1} tomorrow.
- *Definition 2*: A *perfect foresight competitive equilibrium* with land is an infinite sequence of prices p_t and r_t and endogenous variables such that the time t values are a temporary equilibrium satisfying

$$p_{t+1} = p^e_{t+1}$$

- From now on, a perfect foresight competitive equilibrium is simply referred to as an equilibrium.

• Let us derive the rest of the equilibrium conditions for the OLG economy. For simplicity, assume there is no government debt (zero net supply) and that there are no government taxes or transfers.

  - Assume that the asset is initially held by the old. Clearly, only the young would be interested in buying it to hold it until next period
  - The individual budget constraints are then given by

$$\begin{aligned} c^y_t &= \omega^y - p_t a_{t+1} \\ c^o_{t+1} &= \omega^o + (p_{t+1} + d)\, a_{t+1}, \end{aligned}$$

    where a_{t+1} is the amount of the asset purchased by the young in period t.
  - Equilibirum conditions are as follows:
    1. Aggregate savings equals aggregate supply of assets:

$$S^y_t = p_t A$$

       and $a_{t+1} = A$
    2. The interest rate is given by

$$\frac{u'\left(c^o_{t+1}\right)}{u'\left(c^y_t\right)} = 1 + r_{t+1}$$

    3. The price sequence satisfies

$$p_t = \frac{p_{t+1} + d_{t+1}}{1 + r_{t+1}}$$

• Finding an equilibrium:

  1. Guess and verify
     (a) Guess a price p_t and check if the equilibrium conditions are satisfied for the p_{t+1}, p_{t+2}, ... implied by the equilibrium condition, expressed as a combination of the equilibrium conditions:

$$p_t = f_t\left(p_{t+1}, d_{t+1}, A\right)$$

(b) Restrict attention to stationary equilibria where $p_t = p_{t+1}$ is constant over time and thus might be easily guessed at.

2. Solve (numerically) the sequence of prices using the pricing function

- The economy impose some natural restrictions on the price sequence, such as ruling out negative prices or price sequences that are explosive: there typically exists some upper bound on how large prices can be.

- Return to our example economy and look for a stationary equilibrium:

  – Suppose there is a stationary equilibrium where the asset has price p so the price sequence condition becomes

$$p = \frac{p+d}{1+r} \tag{1}$$

  – To clear the market for the asset, the young must buy all of it, so the consumption allocation becomes

$$\begin{aligned} c_t^y &= \omega^y - pA = c^y \\ c_{t+1}^o &= \omega^o + (p+d)\,A = c^o, \end{aligned}$$

  so the interest rate becomes

$$\frac{u'\left(\omega^y - pA\right)}{u'\left(\omega^o + (p+d)\,A\right)} = 1 + r = \frac{p+d}{p} \tag{2}$$

  – Consider two cases:

    1. Land yields some dividends ($d > 0$) and the interest rate is positive ($r > 0$). Then equation (1) becomes

$$p = \frac{d}{r},$$

    i.e., the price is the present value of the future dividends

    2. Land does not yield any dividends ($d = 0$). Then equation (1) becomes

$$p = \frac{p}{1+r}.$$

    This implies two possibilities :

    (a) Autarky: $p = 0$. This gives the same "autarky" equilibrium as we analyzed before (regardless of r and the endowments)

    (b) Bubble: $r = 0$. This implies an Euler equation (2) of

$$\frac{u'\left(\omega^y - pA\right)}{u'\left(\omega^o + pA\right)} = 1,$$

    so that

$$\omega^y - pA = \omega^o + pA,$$

which implies

$$p = \frac{\omega^y - \omega^o}{2A}$$

and equal consumption:

$$c^y = c^o = \frac{\omega^y + \omega^y}{2}.$$

Clearly, this can be an equilibrium only if $p \geq 0$, i.e., only if $\omega^y > \omega^o$ so that the autarky equilibrium is dynamically inefficient (and the autarky interest rate is negative). Note: the asset has a positive price even if it will never pay a dividend. This is a *rational bubble.*

- Lessons:

  1. Rational bubbles can arise only if the interest rate is sufficiently low (lower than the growth rate of the economy)
  2. Bubbles are good: it is an alternative to government debt and pay-as-you-go pensions to deal with dynamic inefficiency.
  3. Bubbles can burst (if people suddenly starts believing in $p = 0$, then the game is over) and this gives a welfare loss

2.2 Pensions and fiscal rules

- Assume the economy is growing at a constant rate γ and that the world-market interest rate is constant at r
- Law of motion for government debt is given by

$$B_{t+1} = (1+r)\, B_t + G_t - T_t$$

and in shares of GDP:

$$\begin{aligned} (1+\gamma)\, b_{t+1} &= (1+r)\, b_t + \underbrace{g_t - \tau_t}_{\text{primary deficit}} \\ &\Rightarrow \\ b_{t+1} &= \frac{1+r}{1+\gamma} b_t + \frac{g_t - \tau_t}{1+\gamma} \end{aligned}$$

- The ratio $\frac{1+r}{1+\gamma}$ determines the drift of debt – interest effect (high r increases debt burden) versus growth effect (high γ alleviates debt burden)
- Suppose the primary deficit and debt are constant over time (as a share of GDP). This implies

$$\begin{aligned} b &= \frac{1+r}{1+\gamma} b + \frac{g - \tau}{1+\gamma} \\ &\Rightarrow \\ b &= \frac{\tau - g}{r - \gamma} \end{aligned}$$

- Suppose, first, that $r < \gamma$ (dynamically inefficient case)
  - Could sustain for ever a defict equal to

$$g - \tau = (\gamma - r)\, b$$

  - No-ponzi scheme condition does not hold!

- Suppose (more reasonably) that $r > \gamma$. In this case we need a perpetual primary surplus to keep debt-output ratio constant, where

$$b = \text{NPV of future primary surpluses}$$

  - so a no-Ponzi scheme condition must hold
  - Note: if you want to run perpetual deficits, it is necessary with $b < 0$

- Consider some examples of relationship between b, $r - \gamma$, and $\tau - g$. Compute the primary surplus required by a particular b and $\gamma - r$. Magnitudes are large:

$b \backslash r - \gamma$	0.5%	4%
50%	0.25%	2%
200%	1%	8%

- Compute the maximum debt that could possibly be sustained. Assume:
  - taxes are at top of Laffer curve, say $\tau = 50\%$ of GDP
  - government spending is a a minimum to run a state (e.g., zero transfers and only basic services), say $g = 10\%$ of GDP

$$\bar{b} = \frac{\tau - g}{r - \gamma} = \frac{0.40 - 0.10}{r - \gamma}$$

$r - \gamma$	0.5%	4%
$\bar{b}$	60	7.5

  Note: souvereign debt issues would kick in *long* before reaching these levels

- Consider now a country that has a large wealth $b < 0$ (due to finding oil, say). How should the oil wealth be distributed across generations? Consider two simple rules:

  1. Rule 1: all generations get the same contribution from the fund (in levels, i.e., kroner). Clearly, to keep B constant it is necessary to take out

$$-r * B$$

     every period. With for example $r = 4\%$, this gives the rule "eat 4% of fund every period"

- Note that with groath ($\gamma > 0$), $B_t/Y_t \to 0$ in the long run
- Motivation: future generations are much richer, so it is fair that the current ones get more as a share of their GDP
- Rule was agreed upon in 2000. Then, long-run real interest rates were 3-4% and with an even higher return to capital (due to a risk premium, say), it seemed conservative to go for a 4% rule
- Current long (30-year) interest rates on debt are low (and have fallen a lot, from 3% to about 1%). Assuming an unchanged risk premium, the rule preserving the size of B should be lower (2-3%, perhaps)

2. Rule 2: all generations get a take-out fom the fund equal to the same share of their GDP
   - Motivation: government services might be produced using workers for which there is little productivity growth (e.g. teachers or the military)
   - Necessary to keep $b_t = B_t/Y_t$ constant
   - The take out (i.e., long-run primary deficit as a share of GDP) is then given by
   $$g - \tau = -(r - \gamma)\, b$$
   With e.g. $r - \gamma = 2\%$ and $-b = 4$ (optimistic view of the Norwegian case), we get
   $$g - \tau = 2\% \cdot 4 = 8\%.$$
   As a share of the value of the fund this becomes
   $$\frac{g - \tau}{b} = r - \gamma = 2\%,$$
   i.e., only half the current rate of extraction.

2.3 More on Pensions

There are two types of pension systems: pay-as-you-go and fully funded systems

- Pay-as-you-go: no accumulation of funds. Every period benefits are paid from current taxes:
  $$\text{benefits} = T$$
  - Crowds out private savings (because households have less disposable income when young and more (pension) income when old
  - Since government savings does not change, it crowds out aggregate savings and, hence lower the capital stock. To see this, note the market-clearing condition for the savings:
  $$S_t = b + k$$
  where b is government debt and k is capital.

- Fully funded system: government saves the pension-tax revenue, so no effect on aggregate savings (increased government savings h matches reduction in private savings)

$$S_t = b + h + k$$

  - Note: no need for such pension system unless some people are irrational (rational households can save on their own)