1 Lecture Notes: The Solow Model

Questions:

1. Why large differences in growth rates?
2. Why persistent differences in productivity?
3. What drives overall world growth?

1.1 The Solow model: theory

- Technology: Firms produce a generi good Y with the production function

$$Y = F(K, AL) = K^{\alpha} (AL)^{1-\alpha}$$

  - L is the number of workers
  - A is the number of "efficiency units" per worker
  - AL is the aggregate number of efficiency units. Assume implicitly that all efficieny units are perfect substitutes)
  - Note that F satisfies constant return to scale and the Inada conditions
  - The generic good can be used for consumption and investment
  - Assume constant growth in L and A:

$$\begin{aligned} L_{t+1} &= (1+n) L_t \\ A_{t+1} &= (1+g) A_t \end{aligned}$$

  - Law of motion for capital:

$$K_{t+1} = (1-\delta) K_t + I_t$$

  - Closed economy

- Preferences: Assume a labor supply of one unit per person and assume a constant savings rate:

$$S_t = sY_t$$

- Markets: Competitive markets for labor, capital, and the consumption/investment good.
- Competitive equilibrium:

1. Firm optimization: prices are marginal productivities,

$$
\begin{aligned}
& \max_{K,L} \{F(K, AL) + (1-\delta) K - wL - (1+r) K\} \\
& \Rightarrow \\
0 &= F_1(K, AL) + 1 - \delta - (1+r) \\
0 &= F_2(K, AL) - w \\
& \Rightarrow \\
r + \delta &= \alpha K^{\alpha-1} (AL)^{1-\alpha} = \alpha \frac{Y}{K} \\
w &= \alpha K^{\alpha} (AL)^{1-\alpha} / L = \alpha \frac{Y}{L}
\end{aligned}
$$

2. Individual optimization: CHEAT by assuming $l = 1$ and $C = (1-s) Y$.
3. Market clearing
   (a) Market for labor:
   (b) Market for capital: Closed economy implies savings equal investments
   $$S_t = I_t$$
   (c) Goods market clearing requires
   $$Y_t = C_t + I_t$$

- Normalize capital and output per efficency unit

$$
\begin{aligned}
y_t &= \frac{Y_t}{A_t L_t} \\
k_t &= \frac{K_t}{A_t L_t}
\end{aligned}
$$

Note that

$$y_t = \frac{K_t^{\alpha} (A_t L_t)^{1-\alpha}}{A_t L_t} = \left(\frac{K_t}{A_t L_t}\right)^{\alpha} = k_t^{\alpha} \equiv f(k_t)$$

- Dynamics of the economy: Note: K_{t+1} is a *stock*, while I_t and capital growth $\Delta K_{t+1} = K_{t+1} - K_t$ are *flows*

$$
\begin{aligned}
\Delta K_{t+1} &= K_{t+1} - K_t = (1-\delta) K_t + I_t - K_t \\
&= sY_t - \delta K_t
\end{aligned}
$$

In efficiency units:

$$
\begin{aligned}
\Delta k_{t+1} &= k_{t+1} - k_t \\
&= \frac{K_{t+1}}{A_{t+1}L_{t+1}} - \frac{K_t}{A_t L_t} \\
&= \frac{K_{t+1}}{A_{t+1}L_{t+1}} - \frac{K_t}{A_{t+1}L_{t+1}} \frac{A_{t+1}L_{t+1}}{A_t L_t} \\
&= \frac{K_{t+1}}{A_{t+1}L_{t+1}} - \frac{K_t}{A_{t+1}L_{t+1}} (1+g)(1+n) \\
&= \frac{K_{t+1} - K_t - K_t\left[(1+g)(1+n) - 1\right]}{A_{t+1}L_{t+1}} \\
&= \frac{sY_t - \delta K_t - K_t\left[(1+g)(1+n) - 1\right]}{A_{t+1}L_{t+1}} \\
&= \frac{sY_t - K_t\left[g + n + \delta + gn\right]}{A_{t+1}L_{t+1}}
\end{aligned}
$$

$$
\begin{aligned}
\Delta k_{t+1} \frac{A_{t+1}L_{t+1}}{A_t L_t} &= \frac{sY_t - K_t\left[g+n+\delta+gn\right]}{A_t L_t} \\
\Delta k_{t+1}(1+g)(1+n) &\approx \underbrace{sf(k_t)}_{\text{gross inv.}} - \underbrace{\left[g+n+\delta\right]k_t}_{\substack{\text{replacement needed} \\ \text{to keep capital constant}}}
\end{aligned}
$$

- Definition: a steady state (with technical progress and population growth) is an equilibrium path in which

$$\Delta k_{t+1} = 0.$$

- Solve for steady state k^*:

$$
\begin{aligned}
0 &= sf(k^*) - \left[g+n+\delta+gn\right]k^* \\
&\Rightarrow \\
\frac{f(k^*)}{k^*} &= \frac{(k^*)^\alpha}{k^*} = \frac{g+n+\delta+gn}{s} \\
k^* &= \left(\frac{s}{g+n+\delta+gn}\right)^{\frac{1}{1-\alpha}}
\end{aligned}
$$

- Consider effect on k^* from changes in savings rate s and in population growth n. GRAPH (y, k graph and time-series)

- **Golden rule**: solve for the maximum steady-state consumption c^{**}:

$$\begin{aligned} C^* &= Y^* - I^* \\ I^* &= (g + n + \delta + gn)\, ALk^* \\ c^* &= \frac{Y^* - I^*}{AL} = \frac{Y^* - (g + n + \delta + gn)\, ALk^*}{AL} \\ &= f(k^*) - (g + n + \delta + gn)\, k^* \\ &\approx f(k^*) - (g + n + \delta)\, k^* \end{aligned}$$

$$\begin{aligned} \frac{\partial c^*}{\partial k^*} &= f'(k^{**}) - (g + n + \delta) = 0 \\ &\Rightarrow \\ f'(k^{**}) &= g + n + \delta \end{aligned}$$

Recall that, from firm's optimization,

$$r_t + \delta = f'(k_t)$$

so requirement for **golden rule** is

$$r^{**} = n + g,$$

i.e., all return to capital is reinvested (zero dividends consumed by capitalists).

1.2 Empirical performance of the Solow model

- Want to evaluate the Solow model empirically. Focus on dynamics (i.e., speed of growth).
- "Neoclassical hypothesis": *all* differences in growth and GDP per capita are due to differences in capital
- Problem: difficult to measure capital. Solution: derive implied growth in Y_t without need to measure K_t:

$$\begin{aligned} Y &= K^\alpha (AL)^{1-\alpha} \\ &\Rightarrow \\ K &= Y^{\frac{1}{\alpha}} (AL)^{-\frac{1-\alpha}{\alpha}} \end{aligned}$$

$$\begin{aligned} K_{t+1} &= sY_t + (1-\delta)\, K_t \\ &\Rightarrow \\ Y_{t+1}^{\frac{1}{\alpha}} (A_{t+1}L_{t+1})^{-\frac{1-\alpha}{\alpha}} &= sY_t + (1-\delta)\, Y_t^{\frac{1}{\alpha}} (A_t L_t)^{-\frac{1-\alpha}{\alpha}} \\ \left(\frac{Y_{t+1}}{L_{t+1}}\right)^{\frac{1}{\alpha}} (A_{t+1})^{-\frac{1-\alpha}{\alpha}} L_{t+1} &= s\left(\frac{Y_t}{L_t}\right) L_t + (1-\delta) \left(\frac{Y_t}{L_t}\right)^{\frac{1}{\alpha}} (A_t)^{-\frac{1-\alpha}{\alpha}} L_t \\ \left(\frac{A_{t+1}}{A_t}\right)^{-\frac{1-\alpha}{\alpha}} \frac{L_{t+1}}{L_t} \left(\frac{Y_{t+1}}{L_{t+1}}\right)^{\frac{1}{\alpha}} &= (A_t)^{\frac{1-\alpha}{\alpha}} s\left(\frac{Y_t}{L_t}\right) + (1-\delta)\left(\frac{Y_t}{L_t}\right)^{\frac{1}{\alpha}} \end{aligned}$$

which boils down to

$$(1+n)(1+g)^{-\frac{1-\alpha}{\alpha}}(\tilde{y}_{t+1})^{\frac{1}{\alpha}} = (A_t)^{\frac{1-\alpha}{\alpha}} s\tilde{y}_t + (1-\delta)(\tilde{y}_t)^{\frac{1}{\alpha}} \tag{1}$$

where $\tilde{y}$ is GDP per capita:

$$\tilde{y}_t = \frac{Y_t}{L_t}$$

1. Assume that all countries have the same $\delta = 5\%$ and $g = 2.5\%$ (same as the US, 1960-2000). Note: $1-\alpha$ is labor's share of output: from firm's optimization we have

$$\begin{aligned} w &= (1-\alpha)\frac{Y}{L} \\ &\Rightarrow \\ 1-\alpha &= \frac{wL}{Y}, \end{aligned}$$

   which is 2/3 for the US $\Rightarrow \ \alpha = 1/3$.

2. Assume that the US is in steady state in 1960. Measure Y_{1960}^{US} and L_{1960}^{US}. Pin down K_{1960}^{US}: and A_{1960}^{US}
3. Measure $\tilde{y}_{1960}^{j} = Y_{1960}^{j}/L_{1960}^{j}$ for a country j.
4. Assume that $A_{1960}^{j} = A_{1960}^{US}$. Use equation (1) to project future values of Y_t^j (note: for the US it is simply

$$\tilde{y}_t^{US} = (1+g)^{t-1960} \cdot \tilde{y}_{1960}^{US}$$

5. Observation: Solow model implies too fast convergence. So a large amount of the differences in output across countries must be driven by differences in A_t^j.

1.3 Conclusion

1. Empirical standpoint: Solow model fails to explain in a satisfactory way the great disparities in output levels and growth rates

2. Theoretical standpoint: Solow model (or simple extensions of it) cannot explain the growth in A_t, which is the main drive of growth

Need "new growth theory" to explain why A_t^j does or does not grow.