Final Exam II (Solutions)
ECON 4310, Fall 2014

1. Do **not write with pencil**, please use a ball-pen instead.
2. Please answer in **English**. Solutions without traceable outlines, as well as those with unreadable outlines **do not earn points**.
3. Please start a **new page** for **every** short question and for every subquestion of the long questions.

Good Luck!

Exercise A:
Short Questions (60 Points)

Answer each of the following short questions on a seperate answer sheet by stating as a first answer True/False and then give a short but instructive explanation. You can write, calculate, or draw to explain your answer. You only get points if you have stated the correct short answer True/False *and* provided a correct explanation to the question. We will not assign negative points for incorrect answers.

Exercise A.1: (10 Points) Static competitive equilibrium

Consider a static economy with a representative consumer that has the following preferences over consumption, c, and labor supply, h,

$$u(c,h) = \log(c) + \log(1-h),$$

and is subject to the budget constraint

$$c = wh,$$

where w is the wage rate per unit of labor supplied. The optimal labor supply is then independent of the wage rate,

$$h = 1/2.$$

True or false?

Your Answer:

True: ☒ False: ☐

Don't forget the explanation!
Reduce consumption in the utility function

$$u(c,h) = \log(c) + \log(1-h) = \log(wh) + \log(1-h),$$

such that the optimality condition with respect to the labor supply, h, reads

$$0 = \frac{1}{wh}w - \frac{1}{1-h} \quad \Leftrightarrow \quad (1-h) = h \quad \Leftrightarrow \quad h = 1/2.$$

Exercise A.2: (10 Points) Solow model, steady-state capital stock

Consider the capital accumulation equation of the Solow model with exogenous technology growth

$$K_{t+1} = sK_t^{\alpha}(A_t L)^{1-\alpha} + (1-\delta)K_t,$$

where K_t is the aggregate capital stock, A_t is the state of technology, L the constant size of the population, s the exogenously given savings (investment) rate, $\alpha \in (0,1)$ the capital income share in the economy, $\delta \in (0,1)$ the depreciation rate of physical capital, $g \geq 0$ denotes the net growth rate of technology, and the capital stock per efficiency unit is defined as

$$k_t \equiv K_t/(A_t L).$$

The stable steady-state capital stock per efficiency unit, $k^\star > 0$, of this Solow model is given by

$$k^\star = \left(\frac{s}{\delta + g}\right)^{1/(1-\alpha)}.$$

True or false?

Your Answer:

True: ☒ False: ☐

Don't forget the explanation!
First, detrend the capital accumulation equation by multplying by $1/(A_t L)$ to yield the accumulation equation in terms of efficiency units

$$(1+g)k_{t+1} = sk_t^{\alpha} + (1-\delta)k_t.$$

In the steady-state, the stable capital stock per efficiency unit will be constant, $k^\star \equiv k_{t+1} = k_t > 0$, such that the capital accumulation equation can be written as

$$(\delta + g)\, k^\star = s(k^\star)^{\alpha} \quad \Leftrightarrow \quad (k^\star)^{1-\alpha} = \frac{s}{\delta + g} \quad \Leftrightarrow \quad k^\star = \left(\frac{s}{\delta + g}\right)^{1/(1-\alpha)}.$$

Thus, the stated steady-state capital stock per efficiency unit is the correct one.

Exercise A.3: (10 Points) Solow model, unexpected shock to technology

Consider the same Solow model described in short question A.2. Suppose the economy is in the stable steady-state and experiences in period t_0 an unexpected major innovation such that the level of technology jumps in period t_0 to $A'_{t_0} > A_{t_0}$, where A_{t_0} denotes the technology level before the shock. Remember that the rental rate in any given period t is given by

$$r_t = \alpha k_t^{\alpha-1}, \quad k_t \equiv K_t/(A_t L).$$

The rental rate will fall on impact, $r'_{t_0} < r_{t_0}$, when the unexpected shock hits, true or false?

Your Answer:

True: ☐ False: ☒

Don't forget the explanation!
On impact the capital stock per efficiency unit falls because the level of technology increases, and this will increase the interest rate. To see this consider the derivative of the interest rate with respect to the technology level

$$\frac{\partial r_t}{\partial A_t} = \alpha(\alpha - 1)k_t^{\alpha-2}\frac{\partial k_t}{\partial A_t} > 0,$$

as $(\alpha - 1)$ is negative and the capital stock per efficiency unit decreases in the level of technology. Thus, the interest rate must jump upwards on impact, as capital is suddenly more productive.

Exercise A.4: (10 Points) Ramsey model, Golden Rule capital stock

Consider the capital accumulation equation of the Ramsey model with exogenous technology growth

$$(1+g)k_{t+1} = k_t^\alpha - c_t + (1-\delta)k_t, \quad k_t \equiv K_t/(A_tL),\, c_t \equiv C_t/(A_tL),$$

where K_t is the aggregate capital stock, A_t is the state of technology, L the size of the population, C_t aggregate consumption, $\alpha \in (0,1)$ the capital income share in the economy, $\delta \in (0,1)$ the depreciation rate of physical capital, and $g \geq 0$ denotes the net growth rate of technology.

The Golden Rule capital stock per efficiency unit (the capital stock per efficiency unit that maximizes steady-state consumption per efficiency unit) is given by

$$k_{GR} = \left(\frac{\alpha\delta}{\delta+g}\right)^{1/\alpha}.$$

True or false?

Your Answer:

True: ☐ False: ☒

Don't forget the explanation!
The Golden Rule capital stock is characterized by

$$k_{GR} = \arg\max_{k\geq 0} k^\alpha - (\delta+g)k,$$

with the associated optimality condition

$$0 = \alpha k_{GR}^{\alpha-1} - (\delta+g) \quad \Leftrightarrow \quad k_{GR} = \left(\frac{\alpha}{\delta+g}\right)^{1/(1-\alpha)}.$$

Thus, the above statement is wrong.

Exercise A.5: (10 Points) Ramsey model, permanent unexpected decrease in β

Consider the dynamic equilibrium equations of the Ramsey model without exogenous growth

$$\frac{c_{t+1}}{c_t} = \left[\beta(1 + \alpha k_{t+1}^{\alpha-1} - \delta)\right]^{1/\theta}, \quad c_t \equiv C_t/(AL),$$
$$k_{t+1} - k_t = k_t^\alpha - c_t - \delta k_t, \quad k_t \equiv K_t/(AL),$$

where K_t is the aggregate capital stock, $\alpha k_{t+1}^{\alpha-1} - \delta$ the interest rate, $A = 1$ is the constant state of technology, $L = 1$ the constant size of the population, C_t aggregate consumption, $\alpha \in (0, 1)$ the capital income share in the economy, $\delta \in (0, 1)$ the depreciation rate of physical capital, $\beta \in (0, 1)$ is the subjective discount factor, and $1/\theta$ the intertemporal elasticity of substitution.

Suppose that the economy is in the steady-state. In response to a permanent and unexpected decrease of the discount factor, β, consumption per efficiency unit, c_t, will jump downwards on impact. True or false?

Your Answer:

True: ☐ False: ☒

Don't forget the explanation!
The household (or the planner) suddenly learns to be less patient than initially expected. Thus, the household currently underconsumes compared to what would be optimal and adjusts the consumption level upwards, to transfer resources from future to the current periods. The level of consumption will jump upwards on impact.

Exercise A.6: (10 Points) Two-period model, substitution and income effect

In the overlapping generations model discussed in class and in the seminar, first period consumption of the household is given by

$$c_1 = \frac{1}{1 + \beta^{1/\theta}(1 + r - \delta)^{1/\theta - 1}} \left(w_1 + \frac{w_2}{1 + r} \right),$$

where $r - \delta$ is the exogenous interest rate, w_t the wage income in period t, c_1 the individual consumption of the household in period 1, $\delta \in (0, 1)$ the depreciation rate of physical capital, $\beta \in (0, 1)$ is the subjective discount factor, and $1/\theta$ the elasticity of intertemporal substitution (EIS). Given that the second period income is zero, $w_2 = 0$, we have seen that the response of first period consumption to a change in the gross interest rate $1 + r - \delta$ is given by

$$\frac{\partial c_1}{\partial(1 + r - \delta)} = -\frac{(1/\theta - 1)\beta^{1/\theta}(1 + r - \delta)^{1/\theta - 2}}{1 + \beta^{1/\theta}(1 + r - \delta)^{1/\theta - 1}} w_1.$$

If the EIS is strictly higher than 1, then the substitution effect dominates and the household decreases first period consumption in response to an increase in the interest rate (an increase in the price of first period relative to second period consumption). True or false?

Your Answer:

True: ☒ False: ☐

Don't forget the explanation!
If the EIS is relatively high, then the consumption choice is relatively price sensitive, such that the substitution effect dominates the income effect. Technically, $1/\theta > 1$ implies that the above derivative is negative.

Exercise B:
Long Question (60 Points)

A life-cycle overlapping generations model

Consider a representative consumer who lives for only two periods denoted by $t = 1, 2$. The consumer is born in period 1 without any financial assets and leaves no bequests or debt at the end of period 2, such that she is subject to the period-by-period budget constraints

$$c_1 + s = w_1$$
$$c_2 = w_2 + (1 + r)s,$$

where s denotes the amount of savings. The consumer's labor income is w_t in each period and her preferences over consumption can be represented by the utility function

$$U(c_1, c_2) = \log(c_1) + \beta \log(c_2), \quad 0 < \beta < 1. \tag{1}$$

For the moment we abstract from the production side of the economy and simply assume that the consumer can borrow and lend consumption across periods at the given real interest rate, $r > 0$. We assume implicitly that the depreciation rate of capital is zero, $\delta = 0$.

(a) (15 Points) Write down the consumer's net present value budget constraint, and show that the optimal consumption in period 1 is given by

$$c_1 = \frac{1}{1 + \beta} \left(w_1 + \frac{w_2}{1 + r} \right).$$

State also the optimal savings.

Solution:

XX Allocation of points: present-value budget constraint and Lagrangean (5 Points), optimality conditions (5 Points), optimal consumption (3 Points), optimal savings (2 Points) XX Substituting out the savings, s, in the period-by-period budget constraint yields the net present value private budget constraint

$$c_1 + \frac{c_2}{1 + r} = w_1 + \frac{w_2}{1 + r}. \tag{2}$$

Maximizing $U(c_1, c_2)$ subject to the lifetime budget constraint in Equation (2) yields the first-order optimality conditions for consumption (let λ denote the Lagrange multiplier on the lifetime budget constraint)

$$0 = c_1^{-1} - \lambda$$
$$0 = \beta c_2^{-1} - \lambda/(1 + r).$$

Combining the two yields the consumption Euler equation

$$c_2/c_1 = \beta(1 + r). \tag{3}$$

Combining Equations (3) and (2) yields

$$c_1 + \frac{\beta(1+r)}{1+r}c_1 = c_1(1+\beta) = w_1 + \frac{w_2}{1+r},$$

which can be reformulated as first period consumption

$$c_1 = \frac{1}{1+\beta}\left(w_1 + \frac{w_2}{1+r}\right).$$

Optimal savings will then be

$$\begin{aligned} s &= w_1 - c_1 \\ &= w_1\left(\frac{1+\beta}{1+\beta} - \frac{1}{1+\beta}\right) - \frac{1}{1+\beta}\frac{w_2}{1+r} \\ &= \frac{1}{1+\beta}\left(\beta w_1 - \frac{w_2}{1+r}\right). \end{aligned}$$

(b) (5 Points) In the above analysis you will have found that the optimal consumption growth over the life-cycle satisfies the Euler equation

$$\frac{c_2}{c_1} = \beta(1+r).$$

What is the elasticity of intertemporal substitution (EIS)

$$\text{EIS} = \frac{\partial \log(c_2/c_1)}{\partial \log(1+r)},$$

of this model specification then?

Solution:

Take logarithms on both sides of the Euler equation to yield

$$\log(c_2/c_1) = \log(\beta) + \log(1+r).$$

Thus the EIS is equal to 1.

(c) (10 Points) Compute the effect of an increase in the gross real interest rate $1+r$ (remember that this corresponds to an increase in the price of c_1 relative to c_2) on first-period consumption c_1. Is this the income, substitution, or wealth effect of the price change and how does your answer relate to the EIS derived in part (b)?

Solution:

The derivative of first-period consumption with respect to the gross interest rate yields (*XX 5 Points XX*)

$$\frac{\partial c_1}{\partial(1+r)} = -\frac{1}{1+\beta}\frac{w_2}{(1+r)^2} < 0.$$

This effect on first period consumption is a pure wealth effect (*XX 2 Points XX*), as with a EIS $= 1$ (or with logarithmic preferences) the substitution and income effect of a relative price change in consumption cancel exactly out (*XX correct argumentation related to part (b), 3 Points XX*).

We now turn from the representative consumer behavior to the economy as a whole. Suppose that this economy is populated by an infinite sequence of overlapping generations that live for two periods. Each generation is of size, L_t, where

$$L_{t+1} = (1+n)L_t,\ n > 0,\ L_0 > 0,$$

and an individual's old-age income is assumed to be zero,

$$w_2 = 0.$$

There is a production sector that combines aggregate physical capital, K_t, and labor, L_t, according to the technology

$$Y_t = F(K_t, Y_t L_t) = K_t^{\alpha}(A_t L_t)^{1-\alpha}, \quad 0 < \alpha < 1,$$

to produce output Y_t. Markets are competitive such that wage rate and the rental rate of capital are given by their marginal product

$$\begin{aligned} w_t &= (1-\alpha)A_t k_t^{\alpha}, \quad k_t \equiv K_t/(A_t L_t), \\ r_t &= \alpha k_t^{\alpha-1}, \end{aligned}$$

and $A_{t+1} = (1+g)A_t$, $g > 0$, $A_0 > 0$. Young agents save by buying unit claims to next period's capital stock, such that capital market clearing requires that the aggregate savings of the young, S_t, corresponds to the next period physical capital stock

$$S_t \equiv s_t L_t = K_{t+1}, \tag{4}$$

where s_t denotes the savings per capita of the current young.

(d) (10 Points) Compute the aggregate savings, S_t, in this economy and use the capital market clearing condition in Equation (4) to characterize the future capital stock K_{t+1} as a function of the current A_t, k_t and L_t. (Hint: if you were not able to solve for individual consumption and savings in part (a), you can assume that a constant fraction of the wage income is saved by each household,

$$s_t = \gamma w_t, \quad 0 < \gamma < 1,$$

to make further progress.)

Solution:

Aggregate savings are given by (only young agents earn a wage, so their lifetime income is simply w_t)

$$S_t = s_t L_t = \frac{\beta}{1+\beta} w_t L_t$$

substituting for the wage rate, the next period aggregate capital stock can be written as

$$K_{t+1} = \frac{\beta}{1+\beta}(1-\alpha)A_t k_t^{\alpha} L_t. \quad (5)$$

Note that $\gamma \equiv \beta/(1+\beta)$.

(e) (10 Points) Derive the law of motion for the capital stock per efficiency unit, k_{t+1} as a function of k_t, sketch it in a diagram with k_{t+1} on the vertical and k_t on the horizontal axis, and mark the stable steady state in the diagram (you do not have to compute the steady state).

Solution:

Multiply Equation (5) by $1/(A_t L_t)$ to yield

$$(1+g)(1+n)k_{t+1} = \frac{\beta}{1+\beta}(1-\alpha)k_t^{\alpha},$$

such that

$$k_{t+1} = \frac{\beta(1-\alpha)}{(1+\beta)(1+g)(1+n)} k_t^{\alpha}.$$

The future capital stock per efficiency unit, k_{t+1}, is a concave function in k_t with $k_{t+1}(0) = 0$. The stable steady state capital stock is characterized by the point of intersection between the $k_{t+1}(k_t)$ function and the 45-degree line where $k_{t+1} = k_t \equiv k^{\star} > 0$.

(f) (10 Points) Suppose the economy is in the stable steady state. Suddenly, in period t_0, due to a natural desaster half of the aggregate capital stock is destroyed. Sketch the dynamics of the capital stock per efficiency unit caused in response to this unexpected shock. Also, in a separate time diagram, sketch the dynamics of the logarithm of the wage rate over time. Be explicit in the diagrams whether a variable falls/increases by more or less than half on impact.

Solution:

In t_0 the capital stock per efficiency unit will jump down by half as half of the aggregate capital stock is destroyed. However, that triggers additional aggregate capital accumulation (due to the high interest rate) until the economy ends up with the same capital stock per efficiency unit as the economy converges back to the steady state.

The log-wage will fall on impact with the capital stock per efficiency unit (although by less than half due to the concavity in k_t). As the economy converges back to the steady state, wages will recover and end up on the same trajectory (remember that there is the trend growth of technology in the wage) as if the shock never had happened. The steady state wage with or without the shock is in both cases is

$$w_t = (1-\alpha)A_t(k^{\star})^{\alpha}.$$

Exercise C:
Long Question (60 Points)

Precautionary savings

Consider a model in which there are two periods ($t = 1, 2$) and a unit mass of identical agents. In period 2 there are two states, denoted by s_G and s_B. The state turns out to be s_G with probability $p \in (0, 1)$ and thus state s_B happens with probability $1 - p$. Each agent receives income e_1 in period 1 and $e_2(s)$ in state $s \in \{s_G, s_B\}$ of period 2, where $e_2(s_G) \geq e_2(s_B)$. All households (you can think of them as a single representative household) have the same preferences over consumption

$$U = \frac{c_1^{1-\gamma}}{1-\gamma} + \beta \mathrm{E}\left[\frac{c_2(s)^{1-\gamma}}{1-\gamma}\right], \tag{6}$$

where $\gamma \geq 0$, $0 < \beta \leq 1$, and E donotes the expectation operator with respect to the state s. All markets are competitive. Households can buy a bond, b, at price 1 in period 1 which pays an (endogenous) interest $1 + r$ in period 2 and is in zero supply. Households start with initial assets of zero, that is they have no bond holdings initially. Note that there is no capital in this economy. There are also no firms (income is obtained by fishing).

(a) (5 Points) Write down the households state-by-state budget constraints for both periods (you can assume the constraints hold with equality).
Solution:

The budget constraints read

$$\begin{aligned} c_1 + b &= e_1 \\ c_2(s) &= e_2(s) + (1+r)b, \quad \forall s \in \{s_G, s_B\}. \end{aligned}$$

(b) (5 Points) Show that the households' constrained optimization problem is equivalent to maximizing the objective function

$$\widetilde{U} = \frac{(e_1 - b)^{1-\gamma}}{1-\gamma} + \beta \mathrm{E}\left[\frac{(e_2(s) + (1+r)b)^{1-\gamma}}{1-\gamma}\right],$$

with respect to the bond holdings, b. (Hint: no proof is required here, just state the procedure of how to derive the above objective function)
Solution:

The households' optimization problem is to maximize U in Equation (6) subject to the period-by-period budget constraints. The objective function $\widetilde{U}$ is derived by simply reducing consumption in U with the period-by-period budget constraints.

(c) (10 Points) Derive the optimality condition for the households' bond holdings and state as well the the bond market clearing condition. What are the implications of the bond market clearing condition for the equilibrium trading of consumption across time?

Solution:

The optimality condition with respect to bond holdings reads (*XX 5 Points XX*)

$$0 = \frac{\partial \widetilde{U}}{\partial b} = -(e_1 - b)^{-\gamma} + \beta \mathrm{E}(e_2(s) + (1+r)b)^{-\gamma}(1+r). \tag{7}$$

The bond market clearing condition is given by (*XX 3 Points XX*)

$$b = 0,$$

as the bond is in zero supply. Thus, because all household's are identical in equilibrium there will be no trade of consumption across time, the interest rate will adjust such that agents consume exactly their income (*XX 2 Points XX*).

(d) (5 Points) Consider the bond market clearing condition derived in part (c), what is the consumption of households in period one, c_1, and in the two states, $c_2(s_G)$ and $c_2(s_B)$, of period 2 then?

Solution:

Since the bond holdings are zero in equilibrium, the consumption in each state will simply be given by the corresponding income in the particular state

$$\begin{aligned} c_1 &= e_1 \\ c_2(s) &= e_2(s), \quad \forall s \in \{s_G, s_B\}. \end{aligned}$$

(e) (10 Points) Use your results from part (c) to show that in equilibrium the gross interest rate of the bond is given by

$$1 + r = \frac{e_1^{-\gamma}}{\beta \mathrm{E}\left[e_2(s)^{-\gamma}\right]} = \frac{e_1^{-\gamma}}{\beta\left[p e_2(s_G)^{-\gamma} + (1-p) e_2(s_B)^{-\gamma}\right]}. \tag{8}$$

Solution:

Rewrite the optimality condition in Equation (7) as (writing out the expectation operator)

$$\frac{(e_1 - b)^{-\gamma}}{\beta\left[p(e_2(s_G) + (1+r)b)^{-\gamma} + (1-p)(e_2(s_B) + (1+r)b)^{-\gamma}\right]} = 1 + r,$$

in equilibrium, where $b = 0$, this expression simplifies to

$$\frac{e_1^{-\gamma}}{\beta\left[p e_2(s_G)^{-\gamma} + (1-p) e_2(s_B)^{-\gamma}\right]} = \frac{e_1^{-\gamma}}{\beta \mathrm{E}\left[e_2(s)^{-\gamma}\right]} = 1 + r.$$

(f) (10 Points) Assume now that $e_1 = 2$ in period 1, $e_2(s_G) = 3$ and $e_2(s_B) = 1$ in period 2, $\beta = 1/2$, $\gamma = 1$ and $p = 1/2$. What is the gross interest rate $1+r$ in equilibrium?
Solution:

The equilibrium gross interest rate is given by

$$1+r = \frac{1/2}{1/2\,[1/2 \times 1/3 + 1/2 \times 1]} = \frac{1}{1/6 + 3/6} = 3/2.$$

(g) (5 Points) Now assume instead that $e_1 = 2$ in period 1, $e_2(s_G) = 2$ and $e_2(s_B) = 2$ in period 2, $\beta = 1/2$, $\gamma = 1$ and $p = 1/2$. What is the gross interest rate $1+r$ in equilibrium?
Solution:

The equilibrium gross interest rate is given by

$$1+r = \frac{1/2}{1/2\,[1/2 \times 1/2 + 1/2 \times 1/2]} = \frac{1}{1/4 + 1/4} = 4/2 > 3/2.$$

(h) (10 Points) Compare the equilibrium interest rate derived in part (g) to the one in part (f). Comment on your results and relate it to the precautionary savings motive that was discussed in class and the seminars.
Solution:

On average, the second period income under both scenarios is equal to the first period income. However, the scenario in part (f) features risk while there is no risk in part (g) concerning the second period income, and the interest rate is falling in the amount of risk (*XX 5 Points XX*). Since the marginal utility of the stated preferences is strictly convex, there is a precautionary savings motive for the households which is reflected by the fact that in the economy with risk (part (f)) the equilibrium interest rate is smaller than in the economy without risk, because the interest rate has to fall to accommodate the precautionary savings motive and to bring the bond market into the zero bond demand equilibrium (*XX 5 Points XX*).