# Problem Set 7:
# Growth Facts and Strategic Sovereign Default

## Exercise 7.1: Some growth facts for mainland Norway

In this problem set we establish some growth facts about mainland Norway and relate them to the theoretical models studied that far. The data analysis and the questions are cooked up such that you can solve them with the program MS Excel which you can access through the Program Kiosk[1] of the University of Oslo.

(a) Visit the website of Statistics Norway (SSB, www.ssb.no) and download (search for the table number indicated below) aggregate quarterly data for mainland Norway on:

- Statbank, Table 09190 (current prices, seasonally adjusted; 1995K1-2015K2): Gross domestic product Mainland Norway, market values (GDP).
- Statbank, Table 09175 (mainland Norway; 1995K1-2015K2): Compensation of employees (Wage income); Employed persons, Employees and self-employed, seasonally adjusted (Employment); Total hours worked for employees and self-employed, seasonally adjusted (TotalHours).

The date 1995K1 denotes the first quarter of the year1995 and 2015K2 the second quarter of the year 2015. After extracting the data from the SSB Statbank, manually rotate the tables (klick the circular arrow in the header) such that each time series variable appears as a column vector (put the time dimension in the stub and the variable in the head). Save the tables in MS Excel format and merge the two data files such that you end up with the raw data of the following structure:

| Date | GDP | Wage income | Employment | TotalHours |
|---|---|---|---|---|
| 1995K1 | 204’936 | 104’784 | 2048,0 | 758,1 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 2015K2 | 652’230 | 363’313 | 2702,3 | 959,1 |

(b) Create two new time series, Productivity = GDP/TotalHours, and Hours = TotalHours/Employment by adding two new columns to your data set.

(c) Let the observation of time-series $x \in \{\text{GDP}, \text{Wages}, ..., \text{Hours}\}$ at date $t$ be denoted by $x_t$. Normalize the gathered macro data by dividing each time series, $x_t$, by its empirical mean and taking the natural logarithm (you can interpret $\hat{x}_t$ as the approximate percentage deviation from the time series mean)

$$\hat{x}_t \equiv \log\left(\frac{x_t}{T^{-1}\sum_{t=1}^{T} x_t}\right),$$

[1] http://www.uio.no/english/services/it/computer/software/servers/kiosk

where $T$ corresponds to the total number of observations within each time series. (Hint: you could do this normalization in a new Excel worksheet, so you can keep the original structure of the data). Plot $\hat{x}_t$ over time for each variable, do the normalized time-series look stationary or do they have a trend? And if so, what will drive the trend?

**Exercise 7.2: A simple model of strategic sovereign default**

Consider the government of a small open economy that issues public debt in the form of one-period discount bonds (that means it issues debt of amount $b$ at price $q$ in the first period and pays back $b$ in the second period to the international investors that bought the government debt in the first period). The government has a planning horizon of two periods and is initially in a recession, such that the first period income, $y_1 < y_2$, is smaller than the one of the second period, $y_2$. In the second period the government can strategically decide to default on the outstanding government debt, $b$, but suffers a stochastic default cost $\phi$ if it does so. The benevolent government's objective function is to maximize lifetime utility

$$U = \log(c_1) + \beta \mathrm{E}\left[\log(c_2) - \phi \mathcal{I}\right], \quad 0 < \beta < 1,$$

subject to the constraints

$$c_1 \leq qb + y_1$$
$$c_2(\mathcal{I}) \leq y_2 - b(1 - \mathcal{I}),$$

where $\mathcal{I}$ is a default indicator, $\mathrm{E}[\cdot]$ is the expectation operator with respect to the default cost $\phi$. The cost $\phi$ is drawn from a uniform distribution with support $[0, \phi_{max}]$, $\phi_{max} < +\infty$, implying that the density function of the default cost is given by

$$f(\phi) = \frac{1}{\phi_{max}}, \quad \phi \in [0, \phi_{max}],$$

and the expectation of a function $g(\phi)$ is given by

$$\mathrm{E}\left[g(\phi)\right] = \int_0^{\phi_{max}} g(\phi) f(\phi) d\phi.$$

For the ease of exposition, we concentrate on debt issuance in the positive range, $b \geq 0$.

(a) We solve this model by backward induction, thus let's first consider the second period. Assume that the realization of the default cost $\phi$ is known when the government decides over default. The government's indirect utility (or the value) in the second period in case of default ($\mathcal{I} = 1$) is then given by

$$V^d(\phi) = \log(y_2) - \phi,$$

while the value of repayment is

$$V^r(b) = \log(y_2 - b).$$

Characterize the threshold realization of the default cost shock, $\Phi(b)$, that makes the government indifferent between repayment and default. What is the government's optimal choice if $\phi < \Phi(b)$? Is the threshold increasing or decreasing in the outstanding debt level $b$?

(b) What is the lowest debt level $b_{max}$ (out of high debt levels) such that the government will always default? Would any investors ever buy public debt at a strictly positive price from this government over and above $b_{max}$?

(c) Now, let's go back to the first period. Compute the probability of a default in the second period

$$\pi(b) \equiv \text{Prob}(\phi < \Phi(b)) \equiv \int_0^{\Phi(b)} \frac{1}{\phi_{max}} d\phi,$$

for a given debt level, $0 \leq b \leq b_{max}$.

(d) Given that the risk-neutral international investors can alternatively invest in a risk-free bond with gross return $1 + r$, what will be the rational equilibrium price of the public debt, $q(b)$? Compute the price of debt at $b = 0$ and $b = b_{max}$. (Hint: the risk-neutral international investors have to be indifferent between investing in the alternative with return $1 + r$ and the expected return of the government's discount bond.)

(e) Given the default threshold $\Phi(b)$, the government chooses the debt level, $b$, to maximize the utility

$$\begin{aligned} U &= \log\left(q(b)b + y_1\right) + \beta \text{E}\left[\log\left(y_2 - b(1 - \mathcal{I})\right) - \phi\mathcal{I}\right] \\ &= \log\left(q(b)b + y_1\right) + \beta\left[\int_{\Phi(b)}^{\phi_{max}} \log(y_2 - b) f(\phi) d\phi + \int_0^{\Phi(b)} \left(\log(y_2) - \phi\right) f(\phi) d\phi\right] \\ &= \log\left(q(b)b + y_1\right) + \beta\left[(1 - \pi(b)) \log(y_2 - b) + \int_0^{\Phi(b)} \left(\log(y_2) - \phi\right) \frac{1}{\phi_{max}} d\phi\right]. \end{aligned}$$

For simplicity, assume that the government takes the default threshold $\Phi(b)$ (and therefore also the default probability $\pi(b)$, and the price $q(b)$) as given such that you can ignore their derivatives with respect to $b$. Derive the first-order optimality condition with respect to $b$, and reformulate it in terms of consumption growth. Give an interpretation of the resulting Euler equation.

(f) Assume that $\beta(1 + r) = 1$. Sketch the optimal consumption path of the government for a given debt level $b$. Consider two different paths with default cost realization, (i) $\phi > \Phi(b)$, and (ii) $\phi < \Phi(b)$.

**Additional Exercises (available on the course website):**

- Exam 2015: A.5.