Problem Set 8:
Real Business Cycles

Exercise 8.1: A two-period real business cycle model

Consider a representative household of a closed economy. The household has a planning horizon of two periods and is endowed with the following preferences over consumption, c, and labor supply, h,

$$U = u(c_1) - v(h_1) + \beta \mathrm{E}\left[u(c_2(s_2)) - v(h_2(s_2))\right],$$

subject to the state-by-state budget constraints

$$\begin{aligned} c_1 + a_2 &= w_1 h_1 \\ c_2(s_2) &= w(s_2)h_2(s_2) + (1+r_2)a_2, \ \forall s_2 \in \mathcal{S} \equiv \{s_G, s_B\}. \end{aligned}$$

The variable s_2 denotes the state of the economy in the second period which follows the stochastic process

$$s_2 = \begin{cases} s_G, \text{ with prob. } p \\ s_B, \text{ with prob. } 1-p, \end{cases}$$

and the household conditions the consumption, $c_2(s_2)$, and the labor supply, $h_2(s_2)$, in the second period on the state, s_2. Note that

$$\mathrm{E}[x(s_2)] \equiv px(s_G) + (1-p)x(s_B),$$

denotes the expected value of any variable x that is a function of the future state of the economy, s_2. In each state of the economy, s_t, there is a linear (in labor n_t) production technology of the form

$$y_t(s_t) = A(s_t)n_t(s_t),$$

such that the competitive wage is given by

$$w(s_t) = \frac{\partial y_t(s_t)}{\partial n_t(s_t)} = A(s_t),$$

where the labor productivity is higher in the good state,

$$A(s_G) = A + (1-p)\sigma > A(s_B) = A - p\sigma, \quad \sigma > 0,$$

than in the bad state of the second period. In the first period, the wage is given by

$$w_1 \equiv w(s_1) = \mathrm{E}\left[w(s_2)\right] = A,$$

such that in expectation both periods yield the same wage (labor productivity). Note that σ is a measure of the risk in the economy as

$$\mathrm{Var}[w(s_2)] = \mathrm{E}\left[w(s_2)^2\right] - \mathrm{E}\left[w(s_2)\right]^2 = p(1-p)\sigma^2.$$

Such that the risk in this economy vanishes as $\sigma \to 0$. We will call that case the deterministic economy.

(a) Let $(\lambda_1, \lambda_2(s_G), \lambda_2(s_B))$ denote the Lagrange multipliers of the state-by-state budget constraints. State the representative agent's Lagrangian.

(b) Derive the optimality conditions with respect to consumption, $(c_1, c_2(s_G), c_2(s_B))$, labor supply, $(h_1, h_2(s_G), h_2(s_B))$ and savings, a_2.

(c) Show that the optimality condition with respect to savings, a_2, can be expressed as the following stochastic consumption Euler equation

$$u'(c_1) = \beta \mathrm{E}\left[u'(c_2(s_2))\right](1 + r_2). \tag{1}$$

Then use the intratemporal optimality conditions to show that

$$w_1 u'(c_1) = v'(h_1), \quad w(s_2)u'(c(s_2)) = v'(h(s_2)), \forall s_2 \in \mathcal{S}. \tag{2}$$

(d) From here onwards we consider the following functional forms for the agent's marginal utility

$$u'(c) = c^{-1}, \text{ (log-utility)}$$
$$v'(h) = h^{1/\varphi}, \varphi > 0.$$

Assuming that the asset a_2 is available in zero supply. Show that the equilibrium return on the asset, $1 + r_2$, which clears the capital market (zero asset demand), $a_2 = 0$, is characterized by the equation

$$(1 + r_2)\beta = \frac{w_1^{-1}}{\mathrm{E}\left[w(s_2)^{-1}\right]} = \frac{u'(w_1)}{\mathrm{E}\left[u'(w(s_2))\right]}.$$

Show that marginal utility $u'(w) = w^{-1}$ is a strictly convex function such that Jensen's inequality applies strictly

$$u'(w_1) = u'(\mathrm{E}[w(s_2)]) < \mathrm{E}[u'(w(s_2))],$$

if $w(s_G) \neq w(s_B)$. If $w(s_G) = w(s_B)$, then the above inequality becomes an equality.

(e) Compare the equilibrium labor supply in the first period, h_1 and the interest rate, $1 + r_2$, of the stochastic economy ($\sigma > 0$) to the equilibrium variables in the deterministic economy ($\sigma = 0$). What is your conclusion, how do the optimal labor supply and the interest rate respond to an increase in risk, σ?

(f) In parts (d) and (e) we have assumed that the asset is given in zero supply. Here we relax this assumption and assume instead that the household faces an exogenous interest rate

$$1 + r_2 = 1/\beta,$$

but may choose nonzero savings at this interest rate. We already know from parts (d) and (e) that in the deterministic economy ($\sigma \to 0$) this interest rate implies zero asset holdings, $a_2 = 0$, and a constant labor supply across periods

$$h_1 = h_2 = 1.$$

Now, let us see what is the optimal labor supply, h_1, and savings, a_2, in the stochastic economy. Suppose we knew the optimal level of savings in the stochastic economy and let denote this level by $\tilde{a}_2$. Show first that the optimal labor supply in the first period, h_1, is increasing in the level of savings, $\tilde{a}_2$. Then show that the optimal level of savings will be strictly positive, $\tilde{a}_2 > 0$. Thus, what is your conclusion about how the optimal labor supply in the first period responds to an increase in risk, σ?

(g) As a last step of the analysis, consider a closed economy where the demand for assets must be equal to the demand for physical capital in the second period

$$a_2 = k_2(r_2).$$

Thus, the demand for capital, k_2 is a decreasing function of the endogenous interest rate, r_2, (think of the interest rate as the marginal product of capital, the higher the capital the lower the marginal product of capital). Intuitively, what will be the response of first-period labor supply, savings, and the capital in this economy if the risk σ increases?

Additional Exercises (available on the course website):

- Exam 2015: A.6, C.
- Retake 2015: C.