Problem Set 9 (Solution Ex. A.2 and B)
ECON 4310, Fall 2015

1. Do **not write with pencil**, please use a ball-pen instead.
2. Please answer in **English**. Solutions without traceable outlines, as well as those with unreadable outlines **do not earn points**.
3. Please start a **new page** for **every** short question and for every subquestion of the long questions.

Good Luck!

	Points	**Max**
Exercise A		**XX**
Exercise B		**XX**
⋮		⋮
Σ		**180**

Grade: __________

Exercise A:
Short Questions (XX Points)

Answer each of the following short questions on a seperate answer sheet by stating as a first answer True/False and then give a short but instructive explanation. You can write, calculate, or draw to reason your answer. You only get points if you have stated the correct True/False *and* provided a correct explanation to the question. We will not assign negative points for incorrect answers.

(XX Points) Exercise A.1: Precautionary savings

Consider the simple two-period real business cycle model discussed in the seminar and in the lecture. With an asset supply of zero, $w_1 = \mathrm{E}[w(s_2)]$, and an optimal consumption profile, $c_1 = w_1$, $c_2(s_2) = w(s_2)$, the stochastic consumption Euler equation in this model is given by

$$\beta(1+r_2) = \frac{u'(c_1)}{\mathrm{E}[u'(c_2(s_2))]} = \frac{u'(\mathrm{E}[w(s_2)])}{\mathrm{E}[u'(w(s_2))]}.$$

The stochastic process for the wage in the second period, $w(s_2)$, takes the form

$$w(s_2) = \begin{cases} w(s_G) = 1 + \sigma/2, & \text{with prob. } 1/2, \\ w(s_B) = 1 - \sigma/2, & \text{with prob. } 1/2, \end{cases}$$

where $\sigma \in (0, 2)$ parametrizes the risk in this economy. Assume that the utility function is of the following form

$$u(c) = c - \frac{1}{4}c^2, \; \forall c \in [0, 2].$$

Does this utility function, $u(c)$, imply precautionary savings?

Your Answer:

Yes: ☐ No: ☐

(XX Points) Exercise A.2: Asset pricing

Let m_{t+1} and x_{t+1} be two random scalars. Then

$$E_t[m_{t+1}x_{t+1}] = E_t[m_{t+1}]E_t[x_{t+1}] + 2Cov_t[m_{t+1}, x_{t+1}],$$

where

$$Cov_t(m_{t+1}, x_{t+1}) \equiv E_t\left[(m_{t+1} - E_t[m_{t+1}])\left(x_{t+1} - E_t[x_{t+1}]\right)\right].$$

Your Answer:

True: ☐ False: ☐

Explanation:

False. Since the covariance is defined as

$$Cov_t(m_{t+1}, x_{t+1}) \equiv E_t\left[(m_{t+1} - E_t[m_{t+1}])\left(x_{t+1} - E_t[x_{t+1}]\right)\right],$$

we can expand the terms within the expectational operator to yield

$$\begin{aligned} Cov_t(m_{t+1}, x_{t+1}) &= E_t\left[m_{t+1}x_{t+1} - m_{t+1}E_t[x_{t+1}] - x_{t+1}E_t[m_{t+1}] + E_t[m_{t+1}]E_t[x_{t+1}]\right] \\ &= E_t[m_{t+1}x_{t+1}] - E_t[m_{t+1}]E_t[x_{t+1}] - E_t[x_{t+1}]E_t[m_{t+1}] + E_t[m_{t+1}]E_t[x_{t+1}] \\ &= E_t[m_{t+1}x_{t+1}] - E_t[m_{t+1}]E_t[x_{t+1}]. \end{aligned}$$

This can be reformulated as

$$E_t[m_{t+1}x_{t+1}] = E_t[m_{t+1}]E_t[x_{t+1}] + Cov_t[m_{t+1}, x_{t+1}],$$

thus the above statement is false.

(XX Points) Exercise A.X: Some more short questions follow ...

Exercise B:
Long Question (XX Points)

Consumption-based asset pricing

Consider a representative household of an economy that maximizes expected lifetime utility

$$U(c_t, c_{t+1}(s_{t+1})) = \frac{c_t^{1-\theta}}{1-\theta} + \beta E_t\left[\frac{c_{t+1}(s_{t+1})^{1-\theta}}{1-\theta}\right], \quad 0 < \beta < 1,$$

by choosing consumption c_t and c_{t+1} in both periods. The agent is endowed with the income e_t and e_{t+1} in the corresponding periods (let e_{t+1} be stochastic), and can shift resources across periods by borrowing or lending in a risk free bond, b_{t+1}, and in a risky asset, a_{t+1}. The risk free bond pays 1 unit of consumption for sure in the future period and has a price q_t in terms of today's consumption, while the risky asset's payoff is the realization of a random variable $x_{t+1} \equiv p_{t+1} + d_{t+1}$ and is of price p_t in terms of today's consumption. The agent initially owns neither assets nor bonds. Expectations are with respect to the future state, $s_{t+1} = (e_{t+1}, x_{t+1}) \in S$, where the set S contains the possible realizations of the state, s_{t+1}.

(a) (XX Points) Derive the agent's budget constraints for both periods (hint: the second period constraint is state-by-state).

Solution:

The agent's budget constraints read

$$c_t = e_t - q_t b_{t+1} - p_t a_{t+1}$$
$$c_{t+1}(s_{t+1}) = e_{t+1} + b_{t+1} + x_{t+1}a_{t+1}, \quad \forall s_{t+1} \in S.$$

(b) (XX Points) Reduce consumption in the utility function using the agent's budget constraints and derive the optimality conditions with respect to bond holdings, b_{t+1}, and asset holdings, a_{t+1}, taking as given prices and returns. (hint: the expectation operator E_t is linear (think of it as a probability weighted sum), so you can just go ahead and take derivatives inside the expectation operator).

Solution:

After reducing consumption, lifetime utility reads

$$U(b_{t+1}, a_{t+1}) = \frac{(e_t - q_t b_{t+1} - p_t a_{t+1})^{1-\theta}}{1-\theta} + \beta E_t\left[\frac{(e_{t+1} + b_{t+1} + x_{t+1}a_{t+1})^{1-\theta}}{1-\theta}\right],$$

and the associated first-order optimality conditions are given by

$$-c_t^{-\theta} q_t + \beta E_t\left[c_{t+1}(s_{t+1})^{-\theta}\right] = 0 \quad \Leftrightarrow \quad q_t = E_t\left[\beta\left(\frac{c_{t+1}(s_{t+1})}{c_t}\right)^{-\theta}\right] \tag{1}$$

$$-c_t^{-\theta} p_t + \beta E_t\left[c_{t+1}(s_{t+1})^{-\theta} x_{t+1}\right] = 0 \quad \Leftrightarrow \quad p_t = E_t\left[\beta\left(\frac{c_{t+1}(s_{t+1})}{c_t}\right)^{-\theta} x_{t+1}\right]. \tag{2}$$

(c) (XX Points) Use the optimality conditions to show that the price of the risky asset is a function of the price of the risk free bond, q_t, and the covariance of the stochastic discount factor, m_{t+1}, with the risky return on the asset,

$$p_t = \beta E_t[m_{t+1}]E_t[x_{t+1}] + \beta Cov_t\left(m_{t+1}, x_{t+1}\right), \quad m_{t+1} \equiv \left(\frac{c_{t+1}}{c_t}\right)^{-\theta}.$$

What factors drive the agent's valuation of the asset?
Solution:

The optimality condition in Equation (2) can be written as

$$p_t = \beta E_t[m_{t+1}]E_t[x_{t+1}] + \beta Cov_t(m_{t+1}, x_{t+1}).$$

Moreover, we know from the optimality condition in Equation (1) that

$$q_t = \beta E_t[m_{t+1}],$$

which gives us the final result that the assets price is given by

$$p_t = q_t E_t[x_{t+1}] + \beta Cov_t(m_{t+1}, x_{t+1}).$$

Thus, the agent's valuation of assets is high, if

(1) the bond price is high (risk free interest is low),
(2) the asset's expected payoff is high,
(3) the payoff is positively correlated to the stochastic discount factor (high payoffs when the marginal utility of consumption in the future period is high (when the endowment e_{t+1} is low), as this provides insurance against future states with low consumption).

Exercise X:
Some More Long Questions ... (XX Points)