Problem Set 5: Overlapping Generations and Pensions

Exercise 5.1: A simple life-cycle overlapping generations model

Consider a representative consumer who lives for only two periods denoted by $t = 1, 2$. The consumer is born in period 1 without any financial assets and leaves no bequests nor debt at the end of period 2. The consumer's labor income is $w_t \geq 0$ in each period and her preferences over consumption can be represented by the utility function

$$U(c_1, c_2) = u(c_1) + \beta u(c_2), \tag{1}$$

where the momentary utility function is given by

$$u(c) = \begin{cases} \frac{c^{1-\theta}-1}{1-\theta}, & \theta \neq 1. \\ \log(c), & \theta = 1, \end{cases}$$

with $\theta > 0$. For the moment we abstract from the production side of the economy and simply assume that the consumer can borrow and lend consumption across periods at the given real interest rate, $r > 0$.

(a) Write down the consumer's net present value budget constraint, and find the optimal consumption and savings over the life-cycle.

(b) Discuss the effect of an increase in the gross real interest rate $1 + r$ (remember that this corresponds to an increase in the price of c_1 in terms of c_2) on first-period consumption c_1. Discuss the income, substitution, and wealth effect of this price change and how they relate to the EIS.

(c) Assume that the consumer has a considerably lower labor income in period 2 compared to period 1 (you can think of period 2 as the old-age period of an agent). To avoid old-age poverty, the government considers to establish a compulsory pension scheme. The pension scheme is supposed to be fully funded, i.e., in period 1 the agent contributes $\tau > 0$ to the pension system and receives $P = (1 + r)\tau$ in the second period. Will the introduction of the pension scheme affect the agent's consumption behavior?

Thinking beyond the simple model under consideration, which circumstances could make such a pension system still worthwhile, and when would it potentially reduce welfare?

(d) Suppose that the contribution to the mandatory pension system is still τ, while the pension in the old-age is $P = (1 + \tilde{r})\tau$, where $\tilde{r}$ reflects the growth rate of the population, n (the biological interest rate). Suppose that the population growth is relatively high, $n > r$, does introduction of this pension system increase the agent's welfare? What kind of pension scheme features indeed a biological interest rate?

We now turn from the behavior of the representative consumer to the economy as a whole. Suppose that this economy is populated by an infinite sequence of overlapping generations that live for two periods. Each generation is of the same size, L, and an individual's old-age income is assumed to be zero, $w_2 = 0$ (this also implies that there won't be any wealth effects). There is no pension scheme, but the remaining specification is as outlined above. There is a production sector that combines aggregate physical capital, K_t, and labor, L, according to the technology

$$Y_t = F(K_t, L) = K_t^{\alpha} L^{1-\alpha},$$

to produce output Y_t. Markets are competitive such that wage rate and the rental rate of capital are given by their marginal product

$$\begin{aligned} w_t &= (1-\alpha)(K_t/L)^{\alpha} \\ r_t &= \alpha(K_t/L)^{\alpha-1}. \end{aligned}$$

Young agents save by buying or selling unit claims to next period's capital stock. We denote today's savings by s_t and you can think it as company shares, for example. Capital market clearing requires that the total savings of today's young corresponds to the next period physical capital stock

$$s_t L = S_t = K_{t+1},$$

where s_t denotes the savings per capita of the current young and S_t the aggregate savings in the economy. Also, note that the depreciation rate of physical capital is set to $\delta = 0$, such that the interest rate and the rental rate are the same and denoted by r_t.

(e) Use this market clearing condition and your previous results to characterize the future capital stock K_{t+1} as a function of the interest rate r_{t+1} and the current capital stock, K_t. Note that savings in period t yield their return in period $t+1$.

(f) Assume that $\theta = 1$ (this implies that the substitution and the income effect cancel out). Sketch the law of motion of the aggregate capital stock in a diagram with K_{t+1} on the vertical and K_t on the horizontal axis, and compute the stable steady-state capital stock of this economy.

(g) Suppose the economy is in the stable steady state. Sketch the dynamics of the aggregate capital stock caused in response to an unexpected and permanent increase in the cohort size by ΔL.

(h) Consider a more general case of the law of motion of physical capital as sketched in Figure 5.3. Mark all steady states of this economy and indicate which of them are locally stable.

(i) Keep assuming that $\theta = 1$. Let the production technology feature labor augmenting technology growth and let the cohort size grow over time

$$Y_t = F(K_t, A_t L_t) = K_t^{\alpha} (A_t L_t)^{1-\alpha},$$

where

$$A_{t+1} = (1+g)A_t,\ g > 0,\ A_0 > 0,$$
$$L_{t+1} = (1+n)L_t,\ n > 0,\ L_0 > 0.$$

Derive the law of motion of the capital stock per efficiency unit, $k_t \equiv K_t/(A_t L_t)$.

(j) Sketch the law of motion of the capital stock per efficiency unit in a diagram with k_t on the horizontal and k_{t+1} on the vertical axis. Suppose the economy is in the stable steady state. Sketch the dynamics of the capital stock per efficiency unit caused in response to an unexpected increase in the cohort size in $t = t_0$ by ΔL (hint: after the shock, which increases L_{t_0} to $L'_{t_0} = L_{t_0} + \Delta L$ the population dynamics continues to follow the general law, $L_{t_0+1} = (1+n)L'_{t_0}$, so the one-time increase has a persistent effect). What are the dynamics of the aggregate capital stock?