Real Business Cycles (Solution)

Exercise: A two-period real business cycle model

Consider a representative household of a closed economy. The household has a planning horizon of two periods and is endowed with the following preferences over consumption, c, and labor supply, h,

$$U = u(c_1) - v(h_1) + \beta \mathrm{E}\left[u(c_2(s_2)) - v(h_2(s_2))\right],$$

subject to the state-by-state budget constraints

$$\begin{aligned} c_1 + a_2 &= w_1 h_1 \\ c_2(s_2) &= w(s_2)h_2(s_2) + (1+r_2)a_2, \ \forall s_2 \in S \equiv \{s_G, s_B\}. \end{aligned}$$

The variable s_2 denotes the state of the economy in the second period which follows the stochastic process

$$s_2 = \begin{cases} s_G, \text{ with prob. } p \\ s_B, \text{ with prob. } 1-p, \end{cases}$$

and the household conditions the consumption, $c_2(s_2)$, and the labor supply, $h_2(s_2)$, in the second period on the state, s_2. Note that

$$\mathrm{E}[x(s_2)] \equiv p x(s_G) + (1-p)x(s_B),$$

denotes the expected value of any variable x that is a function of the future state of the economy, s_2. In each state of the economy, s_t, there is a linear (in labor n_t) production technology of the form

$$y_t(s_t) = A(s_t)n_t(s_t),$$

such that the competitive wage is given by

$$w(s_t) = \frac{\partial y_t(s_t)}{\partial n_t(s_t)} = A(s_t),$$

where the labor productivity is higher in the good state,

$$A(s_G) = A + (1-p)\sigma > A(s_B) = A - p\sigma, \quad \sigma > 0,$$

than in the bad state of the second period. In the first period, the wage is given by

$$w_1 \equiv w(s_1) = \mathrm{E}\left[w(s_2)\right] = A,$$

such that in expectation both periods yield the same wage (labor productivity). Note that σ is a measure of the risk in the economy as

$$\mathrm{Var}[w(s_2)] = \mathrm{E}\left[w(s_2)^2\right] - \mathrm{E}\left[w(s_2)\right]^2 = p(1-p)\sigma^2.$$

Such that the risk in this economy vanishes as $\sigma \to 0$. We will call that case the deterministic economy.

(a) Let $(\lambda_1, \lambda_2(s_G), \lambda_2(s_B))$ denote the Lagrange multipliers of the state-by-state budget constraints. State the representative agent's Lagrangian.

(b) Derive the optimality conditions with respect to consumption, $(c_1, c_2(s_G), c_2(s_B))$, labor supply, $(h_1, h_2(s_G), h_2(s_B))$ and savings, a_2.

(c) Show that the optimality condition with respect to savings, a_2, can be expressed as the following stochastic consumption Euler equation

$$u'(c_1) = \beta \mathrm{E}\left[u'(c_2(s_2))\right](1 + r_2). \tag{1}$$

Then use the intratemporal optimality conditions to show that

$$w_1 u'(c_1) = v'(h_1), \quad w(s_2)u'(c(s_2)) = v'(h(s_2)), \forall s_2 \in S. \tag{2}$$

(d) From here onwards we consider the following functional forms for the agent's marginal utility

$$u'(c) = c^{-1}, \text{ (log-utility)}$$
$$v'(h) = h^{1/\varphi}, \varphi > 0.$$

Assuming that the asset a_2 is available in zero supply. Show that the equilibrium return on the asset, $1 + r_2$, which clears the capital market (zero asset demand), $a_2 = 0$, is characterized by the equation

$$(1 + r_2)\beta = \frac{u'(w_1)}{\mathrm{E}\left[u'(w(s_2))\right]}.$$

Show that marginal utility $u'(w) = w^{-1}$ is a strictly convex function such that Jensen's inequality applies strictly

$$u'(w_1) = u'(\mathrm{E}[w(s_2)]) < \mathrm{E}[u'(w(s_2))],$$

if $w(s_G) \neq w(s_B)$. If $w(s_G) = w(s_B)$, then the above inequality becomes an equality.

(e) Compare the equilibrium labor supply in the first period, h_1 and the interest rate, $1 + r_2$, of the stochastic economy ($\sigma > 0$) to the equilibrium variables in the deterministic economy ($\sigma \to 0$). What is your conclusion, how do the optimal labor supply and the interest rate respond to an increase in risk, σ?

(f) In parts (d) and (e) we have assumed that the asset is given in zero supply. Here we relax this assumption and assume instead that the household faces an exogenous interest rate

$$1 + r_2 = 1/\beta,$$

but may choose nonzero savings at this interest rate. We already know from parts (d) and (e) that in the deterministic economy ($\sigma \to 0$) this interest rate implies zero asset holdings, $a_2 = 0$, and a constant labor supply across periods

$$h_1 = h_2 = 1.$$

Now, let us see what is the optimal labor supply, h_1, and savings, a_2, in the stochastic economy. Suppose we knew the optimal level of savings in the stochastic economy and let denote this level by $\tilde{a}_2$. Show first that the optimal labor supply in the first period, h_1, is increasing in the level of savings, $\tilde{a}_2$. Then show that the optimal level of savings will be strictly positive, $\tilde{a}_2 > 0$. Thus, what is your conclusion about how the optimal labor supply in the first period responds to an increase in risk, σ?

Solution:

From the budget constraint we know that consumption in period zero is given by

$$c_1 = w_1 h_1 - \tilde{a}_2,$$

The intratemporal optimality condition then implies

$$h_1 = w_1^{\varphi} c_1^{-\varphi} = w_1^{\varphi}(w_1 h_1 - \tilde{a}_2)^{-\varphi}.$$

Let's define implicitly the optimal labor supply as a function of $\tilde{a}_2$

$$G(h_1(\tilde{a}_2), \tilde{a}_2) \equiv w_1^{\varphi}(w_1 h_1(\tilde{a}_2) - \tilde{a}_2)^{-\varphi} - h_1(\tilde{a}_2) = 0,$$

such that the implicit function theorem gives us the response of the first period labor supply with respect to the savings $\tilde{a}_2$ as (note that d denotes total derivatives and ∂ partial derivatives)

$$\frac{dG(h_1(\tilde{a}_2), \tilde{a}_2)}{d\tilde{a}_2} = \frac{\partial G(h_1(\tilde{a}_2), \tilde{a}_2)}{\partial h_1(\tilde{a}_2)} \frac{dh_1(\tilde{a}_2)}{d\tilde{a}_2} + \frac{\partial G(h_1(\tilde{a}_2), \tilde{a}_2)}{\partial \tilde{a}_2} = 0,$$

such that

$$\begin{aligned}
\frac{dh_1(\tilde{a}_2)}{d\tilde{a}_2} &= -\frac{\partial G(h_1(\tilde{a}_2), \tilde{a}_2)/\partial \tilde{a}_2}{\partial G(h_1(\tilde{a}_2), \tilde{a}_2)/\partial h_1(\tilde{a}_2)} \\
&= -\frac{w_1^{\varphi}(-\varphi)(w_1 h_1(\tilde{a}_2) - \tilde{a}_2)^{-\varphi-1}(-1)}{w_1^{\varphi}(-\varphi)(w_1 h_1(\tilde{a}_2) - \tilde{a}_2)^{-\varphi-1} w_1 - 1} \\
&= \frac{\varphi w_1^{\varphi}(w_1 h_1(\tilde{a}_2) - \tilde{a}_2)^{-\varphi-1}}{\varphi w_1^{\varphi}(w_1 h_1(\tilde{a}_2) - \tilde{a}_2)^{-\varphi-1} w_1 + 1} \\
&= \frac{\varphi w_1^{\varphi}(w_1 h_1(\tilde{a}_2) - \tilde{a}_2)^{-\varphi}}{\varphi w_1^{\varphi}(w_1 h_1(\tilde{a}_2) - \tilde{a}_2)^{-\varphi} w_1 + (w_1 h_1(\tilde{a}_2) - \tilde{a}_2)} \frac{(w_1 h_1(\tilde{a}_2) - \tilde{a}_2)^{-1}}{(w_1 h_1(\tilde{a}_2) - \tilde{a}_2)^{-1}} \\
&= \frac{\varphi h_1(\tilde{a}_2)}{\varphi h_1(\tilde{a}_2) w_1 + c_1(\tilde{a}_2)} > 0,
\end{aligned}$$

such that household will increase the labor supply in the first period with the level of savings. Note that this relationship between the labor supply and savings is valid independent on whether there is risk in the economy and applies to both cases where the asset demand is allowed to be different from zero in equilibrium.

Can $\tilde{a}_2 = 0$, $h_1 = h_2(s_2) = 1$, be the equilibrium of the stochastic economy? No, the following Euler equation has to be satisfied at the optimal level of savings, $\tilde{a}_2$,

$$\beta(1+r_2) = 1 = \frac{u'(c_1)}{\mathrm{E}\left[u'(c_2(s_2))\right]} = \frac{u'(w_1 h_1 - \tilde{a}_2)}{\mathrm{E}\left[u'(w(s_2)h_2(s_2) + (1+r_2)\tilde{a}_2)\right]},$$

and we have already seen that zero asset holdings are only consistent with an interest rate smaller than $1/\beta$ in the stochastic economy

$$\frac{u'(w_1 h_1 - \tilde{a}_2)}{\mathrm{E}\left[u'(w(s_2)h_2(s_2) + (1+r_2)\tilde{a}_2)\right]} = \frac{u'(w_1)}{\mathrm{E}\left[u'(w(s_2))\right]} < 1 = \beta(1+r_2),$$

by Jensen's inequality. So, in the stochastic equilibrium the marginal utility in the first-period has to be higher (first-period consumption has to be lower) than in the deterministic equilibrium and this is only possible by saving a positive amount, $\tilde{a}_2 > 0$. This also increases the labor supply in the first period $h_1 > 1$ (but only to the extent that consumption is lower and marginal utility higher than before).

In summary, with an exogenously given interest rate $(1+r_2) = 1/\beta$, the household saves more (precautionary savings) and works more in the first period as the risk in the economy increases.

(g) As a last step of the analysis, consider a closed economy where the demand for assets must be equal to the demand for physical capital in the second period

$$a_2 = k_2(r_2).$$

Thus, the demand for capital, k_2 is a decreasing function of the endogenous interest rate, r_2, (think of the interest rate as the marginal product of capital, the higher the capital the lower the marginal product of capital). Intuitively, what will be the response of first-period labor supply, savings, and the capital in this economy if the risk σ increases?

Solution:

The mechanism is very similar to the cases analyzed before: as the risk in the economy increases, the precautionary savings motive of the household increases. Would the interest rate stay unchanged, then the demand for assets exceeds the demand for capital which can not be a capital market equilibrium. Thus, the interest rate will have to fall and this reduces on the one hand the demand for assets and on the other hand it increases the demand for capital (the asset supply!) until the capital market clears again.

The difference to the previous case is that the interest rate will fall less (be higher) compared to the case where the asset supply was fixed, as also the available capital is expanding with the falling interest rate. Also, the increase of the labor supply in the first period will be less pronounced in the first-period, but increase with the amount of savings.