Final Exam (retake)
ECON 4310, Fall 2017

1. Do **not write with pencil**, please use a ball-pen instead.
2. Please answer in **English**. Solutions without traceable outlines, as well as those with unreadable outlines **do not earn points**.
3. Please start a **new page** for **every** short question and for every subquestion of the long questions.

Good Luck!

	Points	**Max**
Exercise A		**40**
Exercise B		**60**
Exercise C		**60**
Σ		**160**

Grade: ___________

Exercise A:
Short Questions (40 Points)

Answer each of the following short questions on a seperate answer sheet by stating as a first answer True/False and then give a short but instructive explanation. You can write, calculate, or draw to explain your answer. You only get points if you have stated the correct True/False *and* provided a correct explanation to the question. We will not assign negative points for incorrect answers.

Exercise A.1: (10 Points) The Optimal Capital Stock

Because more capital allows more output to be produced, it is always better for a country to have more capital stock. True or false?

Your Answer:

True ☐ False: ☒

A per capita capital stock above the golden rule level is so costly to maintain due to depreciation and population growth that reducing the capital stock would ac-tually make it possible to increase consumption in all future periods. The golden rule level is associated with the steady state that maximizes steady-state consumption. Remember that agents in the economy want to maximize their (discounted) utility which depends on consumption, not on maximizing the capital stock.

Exercise A.2: (10 Points) Savings and Economic Growth

An economy that increases its saving rate will experience faster growth. True or false?

Your Answer:

True ☐ False: ☒

True in the short run but not in the long run. In the Solow model an economy that increases its saving rate will temporarily experience faster growth, but the long-run growth rate remains unchanged.

Exercise A.3: (10 Points) Finite horizon model of intertemporal consumption

Consider the optimal intertemporal consumption choice of a household in discrete and finite time $t = 0, 1, \ldots, T < \infty$. The optimal behavior is characterized by the consumption Euler equation

$$\frac{c_{t+1}}{c_t} = [\beta(1+r)]^{1/\theta}, \forall t \leq T,$$

and the private bugdet constraint

$$a_{t+1} + c_t = (1+r)a_t + w, \forall t \leq T,$$

taking as given initial and terminal conditions

$$\begin{aligned} a_0 &= 0 \\ a_{T+1} &= 0, \end{aligned}$$

where a_t denotes asset holdings, r is the exogenous interest rate, w is the constant and exogenously given wage income, c_t the individual consumption of the household, $\beta \in (0, 1)$ is the subjective discount factor, and $1/\theta$ the intertemporal elasticity of substitution.

Suppose that $\beta(1+r) = 0.98$ and $\theta = 1$. Then this household will first accumulate strictly positive assets for a while and then run down assets over the life-cycle. True or false?

Your Answer:

True ☐ False: ☒

The parameter restriction $\beta(1+r) = 0.98$ and $\theta = 1$ implies - through the consumption Euler equation - that consumption will be front loaded over the life-cycle. As the household starts and ends with zero assets, the only way to have the consumption path like that is to first borrow for a while and then repay the debt.

Exercise A.4: (10 Points) Capital and skilled labor complementarity in Solow model

Consider the following version of the Solow model with two types of workers, skilled workers and unskilled workers. The number of workers are given by S and U, respectively. There are no population growth or technology growth. The production function is specified in (**??**), and the economy is described by the following equations

$$K_{t+1} = sY_t + (1-\delta)K_t \tag{1}$$

$$Y_t = \left(\phi_1 \left[\phi_2 K_t^{\frac{\rho-1}{\rho}} + (1-\phi_2) S^{\frac{\rho-1}{\rho}} \right]^{\frac{\rho}{\rho-1}\frac{\sigma-1}{\sigma}} + (1-\phi_1) U^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}, \tag{2}$$

where $0 < \delta < 1$ is the depreciation rate of physical capital, and the saving rate s is constant. $\phi_1, \phi_2\ \rho, \sigma$ are also constant parameters. Denote the share of skilled workers as $\theta \equiv \frac{S}{L} = \frac{S}{S+U}$.

The steady-state level of the capital stock per capita, $k_t \equiv K_t/L$, is described by

$$\delta k^\star = s \left(\phi_1 \left[\phi_2 (k^\star)^{\frac{\rho-1}{\rho}} + (1-\phi_2) \theta^{\frac{\rho-1}{\rho}} \right]^{\frac{\rho}{\rho-1}\frac{\sigma-1}{\sigma}} + (1-\phi_1)(1-\theta)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}.$$

True or false?

Your Answer:

True: ☐ False: ☐

Don't forget the explanation!

In the steady state, we should have $\delta k^\star = sY^\star/L$. Plugging into the production function, note that it is homothetic in the inputs. We can then show

$$\delta k^\star = s \left(\phi_1 \left[\phi_2 (k^\star)^{\frac{\rho-1}{\rho}} + (1-\phi_2) \theta^{\frac{\rho-1}{\rho}} \right]^{\frac{\rho}{\rho-1}\frac{\sigma-1}{\sigma}} + (1-\phi_1)(1-\theta)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}.$$

and the statement in the problem is correct. (the saving rate s is missing in the equation for the regular exam.)

Exercise B:
Long Question (60 Points)
Temporary fiscal shocks in the Ramsey model

Consider a discrete-time version of Ramsey's growth model. The economy is closed and we consider a representative agent with the following preferences over consumption

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t), \tag{3}$$

where c_t denotes period t consumption and $\beta \in (0,1)$ is the subjective discount factor. The momentary utility function is of the form

$$u(c_t) = \frac{c_t^{1-\theta} - 1}{1-\theta},$$

with $\theta > 1$. Every period the agent earns a wage w_t (the labor supply is exogenously set to 1 unit), an interest $r_t a_t$ from her assets holdings and she is subject to the lump-sum tax τ_t. In equilibrium, the agent will choose the sequence consumption and asset holdings $\{c_t, a_{t+1}\}_{t=0}^{\infty}$ to maximize U subject to the period-by-period budget constraint

$$c_t + a_{t+1} = w_t + (1 + r_t)a_t - \tau_t, \tag{4}$$

for a given a_0. The agent is atomic and her decisions do not influence aggregate variables, thus she takes the sequence of taxes, wage rates and interest rates as given.

Firms:
The firm is atomic and acts as a price-taking profit maximizer. Capital can be rented at the rental rate $R_t = r_t + \delta$ (note that the depreciation rate δ is the difference between the rental rate and the interest rate) while labor costs w_t.

The representative firm demands physical capital k_t and labor n_t to produce output y_t with the Cobb-Douglas technology

$$y_t = k_t^{\alpha} n_t^{1-\alpha}. \tag{5}$$

Governments:
The government can raise lump-sum taxes τ_t and rolls over debt in the form of one-period bonds, D_{t+1}, to finance government expenditure, G_t. As it pays an interest rate r_t on the outstanding debt, D_t, the government faces a period-by-period budget constraint

$$G_t = \tau_t + D_{t+1} - (1 + r_t)D_t. \tag{6}$$

Moreover, assume that the time path of government debt is such that it is growing at a lower rate than the interest rate

$$\lim_{T\to\infty} \frac{D_{T+1}}{\Pi_{s=0}^{T}(1 + r_s)} = 0.$$

In other words, it is not feasible for the government to finance the outstanding debt (plus interest payments) by issuing ever more debt as time goes by.

(a) Formulate the Lagrangian of the agent's decision problem (it is common to use λ_t as the Lagrange multiplier on the period t budget constraint).

Derive the first-order conditions for the optimal choice of c_t and a_{t+1}. (15 Points)

Solution:

The Lagrangian of the constrained optimization problem is

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \lambda_t \left[(w_t + (1+r_t)a_t - \tau_t - c_t - a_{t+1} \right]$$

yielding the following first-order conditions for optimal c_t and a_{t+1}

$$0 = \partial\mathcal{L}/\partial c_t = \beta^t u'(c_t) - \lambda_t$$
$$0 = \partial\mathcal{L}/\partial a_{t+1} = -\lambda_t + (1+r_{t+1})\lambda_{t+1}.$$

(b) Derive the Consumption Euler Equation. (5 Points)

Solution:

Eliminate the Lagrange multiplier λ_t by combining the two and derive the Euler equation for consumption

$$\beta^t u'(c_t) = \beta^{t+1} u'(c_{t+1})(1+r_{t+1}) \quad \Leftrightarrow \quad \frac{\beta u'(c_{t+1})}{u'(c_t)} = \frac{1}{1+r_{t+1}}. \tag{7}$$

This is just standard micro theory. The marginal rate of substitution between tomorrow's and today's consumption (the left-hand side) has to be equal to the relative price of tomorrow's in terms of today's consumption (the right-hand side). Relative price interpretation: the agent needs to save $1/(1+r_{t+1})$ units of today's consumption in assets a_{t+1} to yield 1 unit of consumption tomorrow).

(c) Find the first-order conditions for the firm's optimization problem. Show that firms earn zero profits. (10 Points)

Solution:

The firm's optimization problem is static, and it simply maximizes period-by-period profits

$$\pi_t = k_t^{\alpha} n_t^{1-\alpha} - R_t k_t - w_t n_t.$$

First-order conditions with respect to k_t and n_t are

$$0 = \partial\pi_t/\partial k_t = \alpha(k_t/n_t)^{\alpha-1} - R_t$$
$$0 = \partial\pi_t/\partial n_t = (1-\alpha)(k_t/n_t)^{\alpha} - w_t,$$

implying that input factors are paid their marginal product in equilibrium.

(d) Use the government's budget constraint in Equation (**??**) and substitute for D_t iteratively ($t = 1, 2, 3, ...$) to derive the government's intertemporal budget constraint in net present value (NPV) terms

$$D_0 = \sum_{t=0}^{\infty} \frac{\tau_t - G_t}{\Pi_{s=0}^{t}(1 + r_s)}. \tag{8}$$

Give an interpretation of Equation (**??**).

(Hint: you could start out with $D_0 = \frac{1}{1+r_0}[\tau_0 - G_0 + D_1]$, and find a similar expression for D_1 and substitute it into the expression for D_0; do it iteratively.) (10 Points)

Solution:

Just follow the instructions in the problem. Start out with

$$D_0 = \frac{1}{1 + r_0}\left[\tau_0 - G_0 + D_1\right].$$

Then insert for D_1 using the same formula

$$\begin{aligned} D_0 &= \frac{1}{1+r_0}\left[\tau_0 - G_0 + \frac{1}{1+r_1}\left[\tau_1 - G_1 + D_2\right]\right] \\ &= \frac{\tau_0 - G_0}{1 + r_0} + \frac{\tau_1 - G_1}{(1+r_0)(1+r_1)} + \frac{D_2}{(1+r_0)(1+r_1)} \\ &= \sum_{t=0}^{1} \frac{\tau_t - G_t}{\Pi_{s=0}^{t}(1+r_s)} + \frac{D_{1+1}}{\Pi_{s=0}^{1}(1+r_s)}, \end{aligned}$$

and continue until period T to get

$$D_0 = \sum_{t=0}^{T} \frac{\tau_t - G_t}{\Pi_{s=0}^{t}(1+r_s)} + \frac{D_{T+1}}{\Pi_{s=0}^{T}(1+r_s)}.$$

Finally, let $T \to \infty$ to yield Equation (**??**). Thus, the NPV of government expenditures cannot exceed the NPV of lump-sum taxes net of the initial debt position.

Dynamics:

In this economy we know that the solution to the social planner's problem is equivalent to the competitive market equilibrium. According to the social planner's solution, the same consumption Euler equation and resource constraint (goods market clearing) along with the so-called transversality condition (which stands in for the no-Ponzi condition)

$$\begin{aligned} \frac{c_{t+1}}{c_t} &= \left[\beta(1 + r_{t+1})\right]^{1/\theta} = \left[\beta(1 + \alpha k_{t+1}^{\alpha-1} - \delta)\right]^{1/\theta} \\ k_{t+1} - k_t &= k_t^{\alpha} - \delta k_t - c_t - G_t \\ \lim_{t\to\infty} \beta^t c_t^{-\theta} k_{t+1} &= 0 \end{aligned}$$

determine the optimal solution of the dynamic system. Let us assume that $G_t = G$, then we can define two correspondances. One which characterizes all possible combinations of (c_t, k_t) when consumption is constant,

$$\mathcal{C}_1(k) \equiv \left\{ c \in [0, \infty) : c_{t+1}/c_t = \left[\beta(1 + \alpha k^{\alpha-1} - \delta)\right]^{1/\theta}, c_{t+1} = c_t = c \right\},$$

and one which captures all combinations if the physical capital stock is constant,

$$\mathcal{C}_2(k) \equiv \{c \in [0, \infty) : c = k_t^a - (k_{t+1} - (1 - \delta)k_t) - G, k_{t+1} = k_t = k\}.$$

(e) Draw the two correspondances, $\mathcal{C}_1(k)$ and $\mathcal{C}_2(k)$, in a diagram with k on the horizontal axis and c on the vertical axis, the so called phase diagram. (10 Points)

Solution:

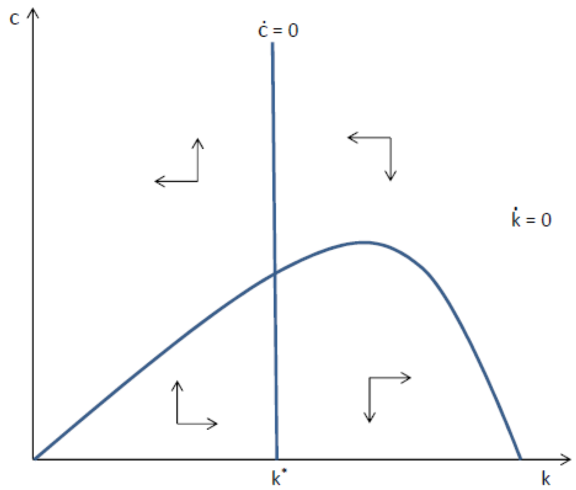

Figure 1: Correspondances for $\mathcal{C}_1(k)$ and $\mathcal{C}_2(k)$

(f) Now assume that the economy has run for a long time and is in its steady state with constant government expenditures, $G_t = G$, and tax policy, $\tau_t = \tau$.

Consider an unexpected and temporary increase of ΔG in government expenditures from period t_0 until period t_1. Sketch the dynamics of consumption and physical capital back to the steady-state in the phase diagram. (10 Points)

Solution:

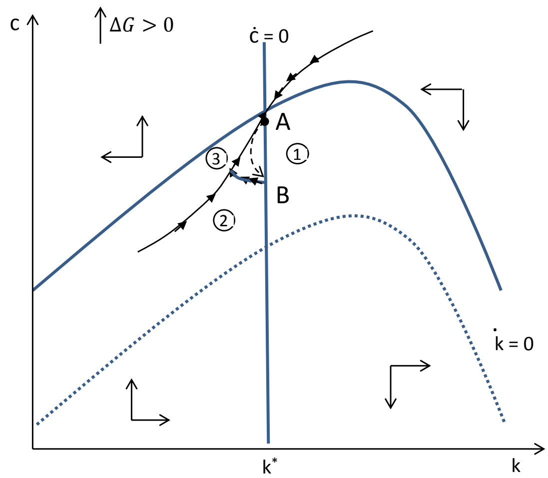

FIGURE 3.1

Figure 2: Dynamics of consumption and physical capital

Exercise C:
Long Question (60 Points)
Labor supply and Real business Cycle Model

Consider a real business cycle model with only technology shocks. The representative household is endowed with one unit of time each period and has expected lifetime utility of,

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\ln(c_t) + \frac{(1-l_t)^\theta}{\theta} \right], \quad 1 > \theta > 0, \tag{9}$$

where c is per capita consumption and l is the amount of time spent working. Assume $0 < \theta < 1$. Capital depreciates at rate δ. Denote the real interest rate by r_t, and the wage rate by w_t. Assume that labor income is taxed at rate τ_t per period, and the tax revenues are rebated in a lump-sum fashion to the household. Assume that output is produced according to a Cobb-Douglas production function,

$$Y_t = K_t^\alpha [A_t L_t]^{(1-\alpha)}, \tag{10}$$

where the technology term follows:

$$\ln A_t = \bar{A} + gt + \tilde{A}_t. \tag{11}$$

The stochastic component of technology $\tilde{A}_t$ is assumed to follow an autoregressive process of order one. Markets are competitive. The household owns the capital of the firm, and decides how much to invest each period, I_t. This is equivalent to deciding how much capital, K_{t+1}, to save for next period. The household earns capital income: $R_t K_t = (r_t + \delta) K_t$

(a) (10 Points) Set up the representative firms problem, find the first-order conditions, and provide an economic interpretation.

Solution:

The firm chooses factor inputs K_t and L_t to maximize profit at each date t:

$$K_t^\alpha [A_t L_t]^{(1-\alpha)} - w_t L_t - (r_t + \delta) K_t$$

The firm increases its demand for a factor until the marginal product equals the factor price, which yields the first order conditions:

$$\alpha K_t^{\alpha-1} (A_t L_t)^{(1-\alpha)} = r_t + \delta$$
$$(1-\alpha) K_t^\alpha A_t^{(1-\alpha)} L_t^{-\alpha} = w_t$$

(b) (10 Points) Write down the household's problem and derive its first order necessary conditions. Hint: We know the variables today with certainty, i.e. $E[x_t] = x_t$ but future variables are uncertain.

Solution:

The Household's problem can be formulated as follows:

$$\begin{aligned} & \max_{c_t, l_t, K_{t+1}} E_0 \sum_{t=0}^{\infty} \beta^t \left[\ln(c_t) + \frac{(1-l_t)^\theta}{\theta} \right] \\ s.t. \quad & c_t + I_t = w_t l_(1-\tau_t) + R_t K_t + T_t \\ & K_{t+1} = (1-\delta) K_t + I_t \end{aligned}$$

where T_t is the lump-sum tax, and $R_t = r_t + \delta$ is the gross rental rate for firms. We can write down the Lagrangian as follows:

$$\begin{aligned} \mathcal{L} &= E_t \sum_{t=0}^{\infty} \beta^t \left[\ln(c_t) + \frac{(1-l_t)^\theta}{\theta} \right] \\ &+ E_t \sum_{t=0}^{\infty} \lambda_t \times \left[w_t l_t (1-\tau_t) + R_t K_t + T_t + (1-\delta) K_t - c_t - K_{t+1} \right], \end{aligned}$$

Now take the first-order conditions, for c_t, l_t and for K_{t+1}, we should have

$$\begin{aligned} \frac{1}{c_t} - \lambda_t &= 0 \\ -(1-l_t)^{\theta-1} + \lambda_t w_t (1-\tau_t) &= 0 \end{aligned}$$

and

$$\begin{aligned} -\lambda_t + E_t \beta \lambda_{t+1} (R_{t+1} + 1 - \delta) &= 0, \\ \text{or } -\lambda_t + \beta E_t \lambda_{t+1} (r_{t+1} + 1) &= 0 \end{aligned}$$

(c) (10 Points) Derive three relationships: (i) between consumption today and consumption tomorrow, (ii) between leisure today and leisure tomorrow, (iii) between consumption today and leisure today.

Solution:

Combining the first and the third FOC, the second and the third and the first and the second you get respectively:

for consumption Euler equation:

$$\frac{1}{c_t} = \beta E_t \frac{1}{c_{t+1}} (r_{t+1} + 1)$$

for inter-temporal labor supply:

$$\begin{aligned} 1 &= \beta E_t \lambda_{t+1}(r_{t+1}+1)/\lambda_t \\ &= \beta E_t \frac{(1-l_{t+1})^{\theta-1}/\left[w_{t+1}(1-\tau_{t+1})\right]}{(1-l_t)^{\theta-1}/\left[w_t(1-\tau_t)\right]}(r_{t+1}+1) \\ &= \beta E_t \frac{(1-l_{t+1})^{\theta-1}}{(1-l_t)^{\theta-1}} \frac{\left[w_t(1-\tau_t)\right]}{\left[w_{t+1}(1-\tau_{t+1})\right]}(r_{t+1}+1) \end{aligned}$$

and lastly for intra-temporal labor supply:

$$\begin{aligned} -(1-l_t)^{\theta-1} + \lambda_t w_t(1-\tau_t) &= 0 \Rightarrow \\ (1-l_t)^{\theta-1} &= \lambda_t w_t (1-\tau_t) \\ &= \frac{1}{c_t} w_t(1-\tau_t). \end{aligned}$$

(d) (10 Points) Find the equations that describe labor supply and demand.

Solution:

Labor demand is given by the firm's optimization problem with respect to Labor:

$$L_t^D = \left(\frac{(1-\alpha)K_t^\alpha A_t^{1-\alpha})}{w_t}\right)^{\frac{1}{\alpha}}$$

While labor supply is given by households optimization problem:

$$L_t^S = \left(\frac{1}{c_t} w_t(1-\tau_t)\right)^{\frac{1}{\theta-1}} - 1$$

(e) (10 Points) How is the current level of consumption affected if the covariance between $1/c_{t+1}$ and r_{t+1} increases. Explain.

Solution:

We can rewrite the Euler equation as,

$$\frac{1}{c_t} = \beta\left\{E_t \frac{1}{c_{t+1}} E_t\left(1+r_{t+1}\right) + Cov\left(\frac{1}{c_{t+1}}, 1+r_{t+1}\right)\right\}$$

If the covariance between $1/c_{t+1}$ and r_{t+1} increases, then current consumption, c_t, has to decrease.

(f) (10 Points) Consider the deterministic version of the model (no uncertainty). How does the relative choice of labor supply between the two periods depend on the relative taxes between periods? Provide intuition for your answer. Do we have Richardian Equivalence.

Solution:

Now the intratemporal optimality condition reads:

$$\frac{(1-l_t)^{\theta-1}}{(1-l_{t+1})^{\theta-1}} = \beta \frac{[w_t(1-\tau_t)]}{[w_{t+1}(1-\tau_{t+1})]}(r_{t+1}+1)$$

If taxes go up in period t, it becomes relatively less atractive to work in that period. The household consumes relatively more leisure and works relatively less in period t compared to period $t+1$. Richardian Equivalence requires lumpsum taxes. In this problem we have distortive taxes on labor income.