Exercise A:

A Simple Life-cycle Model: Warming up (20 points)

Sunny and Cher both obey a two period life-cycle model of consumption. Both starts out with \$100, which they have inherited from their Grandma. Sunny earns \$100 in the first period, while Cher earns \$50. In the second period Cher, who is a college graduate, earns 3 times more than Sunny. Both can borrow and lend at interest rate r. You observe that Sunny and consumes \$150 in both periods while Cher consumes \$200 in the first period and \$230 in the second period. What is the interest rate r?

Solution
The intertemporal budget constraint must hold for both Sunny and Cher. Thus:

$$\begin{array}{ll} \text{Sunny:} & 100 + 100 + \frac{y_2}{(1+r)} = 150 + \frac{150}{(1+r)} \\ \text{Cher:} & 100 + 50 + \frac{3y_2}{(1+r)} = 200 + \frac{230}{(1+r)} \end{array}$$

This is two equations with two unknowns, y_2 and r. Solving, we get $y_2 = 95$ and $r = 10\%$.

Exercise B:

A Simple Life-cycle Model (100 points)

Joe Six-pack just learned about our two period life cycle model and fell so deeply in love with it that he decides that he wants to live (i.e. consume) according to it. He is in the first period of his life and he spends it in college. In college he works in the cafeteria where he makes $20,000 after taxes. He knows for sure that he is going to get a job at McKinsey (he is an economics major) and will make $110,000 after taxes in the second period of his life (sadly enough he only lives for two periods). He has no initial wealth and he can borrow and lend at an interest rate of 10%. His utility function is

$$U(c_1, c_2) = 2\sqrt{c_1} + \sqrt{c_2}$$

(a) (10 Points) Determine Joe's optimal consumption in both periods of his life. Is Joe a saver or a borrower? Determine his optimal level of saving/borrowing.

Solution:

The intertemporal budget constraint of Joe is given by

$$\begin{aligned} c_1 + \frac{c_2}{1.1} &= 20,000 + \frac{110,000}{1.1} \\ &= 120,000 \end{aligned}$$

Maximizing utility subject to the budget constraint yields $c_1 = \$94,117.65$ and $c_2 = \$28470.59$. Joe saves $\$20,000 - \$94,117.65 = -\$74,117.65$. He is a borrower.

(b) (10 Points) How would Joe's optimal consumption and saving decision change if he had a utility function of the form $U(c_1, c_2) = \min\{c_1, c_2\}$? This is a so called Leontief utility function.

Solution:

With this utility function Joe sets $c_1 = c_2 = c$. From the budget constraint we get $c = 62,857.14$

(c) (20 Points) Back to Joe's original utility function for the rest of the question. Suppose the interest rate increases to 16%. Now what is Joe's optimal consumption and saving/borrowing plan? Discuss the change in c_1 and c_2 in light of the income, substitution and human wealth effects.

Solution:

Now the intertemporal budget constraint is given by

$$\begin{aligned} c_1 + \frac{c_2}{1.16} &= 20,000 + \frac{110,000}{1.16} \\ &= 114,827.59 \end{aligned}$$

Maximizing utility subject to the new budget constraint yields $c_1 = 89,013.64$ and $c_2 = \$29,944.19$. He borrows \$69013.64. Since Joe is a borrower, income, substitution and wealth effects are all negative for first period consumption and therefore first period consumption declines with an increase in the interest rate. For second period consumption the income and wealth effects are negative and the substitution effect is positive, so on theoretical grounds it is hard to say whether second period consumption goes up or down. In this example the substitution effect dominates the other effects and consumption in the second period goes up slightly.

(d) (10 Points) Is Joe better off or worse off after the interest rate increase? Justify your answer and provide some economic intuition.

Solution:

The most obvious way to answer this question is to calculate his utility. Before the interest rate increase his optimal consumption was $(c_1, c_2) = (94117.65, 28470.59)$.This gives utility

$$\begin{aligned} U(94117.65, 28470.59) &= 2\sqrt{94117.65} + \sqrt{28470.59} \\ &= 782.31 \end{aligned}$$

After the interest change we have $(c_1, c_2) = (89013.64, 29944.19)$ and hence utility

$$\begin{aligned} U(89013.64, 29944.19) &= 2\sqrt{89013.64} + \sqrt{29944.19} \\ &= 769.75 \end{aligned}$$

Hence Joe is worse off after the interest rate hike. This is basically due to the fact that he is a borrower, facing a higher interest rate for his loan than before.

(e) (10 Points) Suppose the interest rate is back at 10%. The government wants to cut taxes to stimulate the economy. Suppose the government cuts the taxes that Joe has to pay in the first period by $\$2,000$, so that Joe's after tax income increases from $\$20,000$ to $\$22,000$. In order to finance this tax cut the government has to increase taxes in the second period by $\$2000 * (1 + r) = \$2,200$. Hence Joe's income in the second period goes down from $\$110,000$ to $\$107,800$. Was is Joe's new optimal consumption and saving plan. Compare to your answer in Part (a).

Solution:

This is easy! Write down Joe's intertemporal budget constraint

$$\begin{aligned} c_1 + \frac{c_2}{1.1} &= \$22,000 + \frac{\$107,800}{1.1} \\ &= \$120,000 \end{aligned}$$

This is exactly the same budget constrain as in part 1. Since Joe's utility function is the same as in question 1, his optimal consumption is obviously the same as in question 1; it is unaffected by the tax change. What happens to savings. In question 1 we saw that Joe borrowed $\$74,117.65$. Now he borrows $\$94,117.65 - \$22000 = 72,117.65$. Hence Joe's savings go up (his borrowings decline) by exactly the amount by which the tax declines.

(f) (10 Points) Now suppose that Joe in borrowing constrained, in particular assume that he can't borrow at all in the first period of his life. What is the optimal consumption plan now (incomes in both periods are at their original levels)?

Solution:

Joe wants to borrow (see part (a)), but now he can't. So his optimal consumption with borrowing constraints is $c_1 = 20,000$ and $c_2 = 110,000$.

(g) (10 Points) Joe is still borrowing constrained, and now the government carries out the same tax change as in (e). Compute Joe's new optimal consumption plan and compare to the results in the previous question. Explain.

Solution:

Joe is still borrowing constrained, but now the constraint has been somewhat lifted. His optimal consumption is now $c_1 = 22000$ and $c_2 = 107800$ and thus changes, since with borrowing constraints not only the present discount value of lifetime income matters.

(h) (20 Points) Suppose more generally that Joe's utility is

$$U(c_1, c_2) = \frac{c_1^{1-\sigma}}{1-\sigma} + \beta \frac{c_2^{1-\sigma}}{1-\sigma}$$

$\sigma > 0$.He has no initial savings, income y_1 in period 1 and y_2 in period 2. The interest rate is r. What can you say about the effect of a change in the interest rate on savings, $\frac{ds}{dr}$, and consumption in the first period, $\frac{dc_1}{dr}$? How does this depend on the parameter σ? Relate your answer to the income, substitution and human wealth effect.

Solution:

Solving for the optimal consumption path we have:

$$c_1 = (1 + \beta^{\frac{1}{\sigma}}(1+r)^{\frac{1}{\sigma}-1})^{-1}(y_1 + \frac{y_2}{1+r})$$

Hence:

$$\begin{aligned} \frac{\partial c_1}{\partial(1+r)} &= -\frac{y_2}{(1+r)^2}(1 + \beta^{\frac{1}{\sigma}}(1+r)^{\frac{1}{\sigma}-1})^{-1} - (y_1 + \frac{y_2}{(1+r)})(1 + \beta^{\frac{1}{\sigma}}(1+r)^{\frac{1}{\sigma}-1})^{-1} * \\ & (\beta^{\frac{1}{\sigma}}(\frac{1}{\sigma} - 1)(1+r)^{\frac{1}{\sigma}-2}) \end{aligned}$$

There are 3 cases: 1) If $\sigma = 1$, Income and substitution effect cancel out and period 1 consumption declines (savings increases) because of the human capital effect. If $y_2 = 0$, then an increase in the interest rate has no effect the intertemporal consumption allocation. 2) $0 < \sigma < 1$, then the substitution effect dominates the income effect and hence period 1 consumption declines (savings increases). 3) $\sigma > 1$, Then the income effect dominates the substitution effect and the total effect of an increase in the interest rate on $s = y_1 - c_1$ is ambiguous.
