Final Exam
ECON 4310, Fall 2018

1. Do **not write with pencil**, please use a ball-pen instead.
2. Please answer in **English**. Solutions without traceable outlines, as well as those with unreadable outlines **do not earn points**.
3. Please start a **new page** for **every** short question and for every subquestion of the long questions.

Good Luck!

	Points	**Max**
Exercise A		**40**
Exercise B		**60**
Exercise C		**60**
Σ		**160**

Grade: ___________

Exercise A:
Short Questions (40 Points)

Answer each of the following short questions on a separate answer sheet. You will only get points for correct answer with an explanation.

Exercise A.1: (10 Points) Static Competitive Equilibrium

Consider a static economy with a representative consumer that has the following preferences over consumption, c, and labor supply, h,

$$u(c,h) = \log(c) + \log(1-h),$$

and is subject to the budget constraint

$$c = wh,$$

where w is the wage rate per unit of labor supplied. The optimal labor supply is then independent of the wage rate,

$$h = 1/3.$$

True or false?

Your Answer:

True: ☐ False: ☒

Don't forget the explanation! Reduce consumption in the utility function

$$u(c,h) = \log(c) + \log(1-h) = \log(wh) + \log(1-h),$$

such that the optimality condition with respect to the labor supply, h, reads

$$0 = \frac{1}{wh}w - \frac{1}{1-h} \quad \Leftrightarrow \quad (1-h) = h \quad \Leftrightarrow \quad h = 1/2.$$

Exercise A.2: (10 Points) Fiscal Policy

Consider a small open economy populated by non-overlapping generations living one period and by an infinitely lived government, endowed with asset $a_0 = A > 0$ at time $t = 0$. Each generation is subject to the private budget constraint:

$$c_t = w_t + T_t$$

and the government is subject to the period-by-period gov. budget constraint:

$$a_{t+1} = (1+r)a_t - T_t$$

where c_t is private consumption, w_t is exogenous private income, a_t is government's net saving, T_t is a public transfer from the government to the generation t, and r is the constant interest rate.

Wage grows at rate g such that $w_{t+1} = (1+g)w_t$.

The government follows the following fiscal rule:

$$a_t = (1+x)^t A \quad \forall t.$$

If $r < g$ then transfers T_t are negative $\forall t$. True or false?

Your Answer:

True: ☐ False: ☒

Don't forget the explanation!
Solution:

Plug-in the fiscal rule into the Gov. Budget constraint, and solve for T_t:

$$T_t = (1+x)^t A(r-x) \quad \forall t,$$

so that transfers are independent of g and always positive if $r > x$ and negative if $r < x$.

Exercise A.3: (10 Points) Permanent Technology shocks in Real business cycle model and Consumption Response

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Consider a simple two-period model of labor supply, as we saw in lectures, where we assume that utility is separable in consumption and labor supply:

$$\begin{aligned}
\max_{\{c_0,c_1,h_0,h_1,a_1\}} \; & \log c_0 - \phi\frac{h_0^{1+\theta}}{1+\theta} + \beta[\log c_1 - \phi\frac{h_1^{1+\theta}}{1+\theta}] \\
s.t. & \\
c_0 + a_1 &= w_0h_0 + (1+r_0)a_0 \\
c_1 &= w_1h_1 + (1+r_1)a_1
\end{aligned}$$

for given $a_0 = 0$. Assume r_0, r_1 are exogenously given. We know the household has the following intertemporal labor supply condition:

$$\beta\frac{\phi h_1^{\theta}}{\phi h_0^{\theta}} = \frac{w_1}{(1+r_1)w_0},$$

and the solution for h_0 is given by:

$$\phi h_0^{1+\theta}\left[1+\left(\frac{w_1}{(1+r_1)w_0}\right)^{1+\frac{1}{\theta}}\beta^{-\frac{1}{\theta}}\right] = (1+\beta).$$

Suppose there is a permanent change to wages at the beginning of time 0: both wages in the first and second period increase by 10%. Then this household will take advantage of this opportunity and increase consumption in period 0, c_0, by 10%. True or false?

Your Answer:

True ☒ False: ☐

By inspecting the solution for h_0 we know h_0 will not change; Then by using the intertemporal labor supply condition, we know h_1 and h_0 is proportional to each other. Therefore, labor supply h_1 does not change.

Going to consumption and saving, by looking at the Euler equation and the life-time budget constraint:

$$\frac{c_1}{c_0} = \beta(1+r_1), c_0 + \frac{c_1}{1+r_1} = w_0h_0 + \frac{w_1h_1}{1+r_1},$$

we know c_0 and c_1 will both increase by 10%.

Exercise A.4: (10 Points) Optimal policy, Laffer curve

Suppose the aggregate labor supply, $h(\tau)$, of an economy as a function of the labor income tax rate, τ, is given by

$$h(\tau) = [(1-\tau)w]^{1/2}.$$

The top of the Laffer curve is given by $\bar{\tau} = 1/2$. True or false?

Your Answer:

True: ☐ False: ☒

Don't forget the explanation!

False. The top of the Laffer curve is characterized by

$$\bar{\tau} = \arg\max_{0\leq\tau\leq 1} \tau h(\tau) w$$

with the associated optimality condition

$$\begin{aligned} 0 &= [(1-\bar{\tau})w]^{1/2} w + \frac{1}{2}\bar{\tau}\left[(1-\bar{\tau})w\right]^{1/2-1} w(-w) \\ &= 1 - \frac{1}{2}\bar{\tau}\left[(1-\bar{\tau})w\right]^{-1} w, \end{aligned}$$

such that the top of the Laffer curve is given by

$$2 = \frac{\bar{\tau}}{1-\bar{\tau}} \quad \Leftrightarrow \quad \bar{\tau} = 2/3.$$

Alternatively, along the notation in the seminar, setting the Frisch elasticity of labor supply to $\varphi = 1/2$, and the tax elasticity of the labor supply is given by $e(\tau) = \varphi\tau/(1-\tau)$. The top of the Laffer curve satisfies $e(\bar{\tau}) = 1$ whih yields the same value, $\bar{\tau} = 2/3$.

Exercise B:
Long Question (60 Points)

OLG

Consider a representative consumer who lives for only two periods denoted by $t = 1, 2$. The consumer is born in period 1 without any financial assets and leaves no bequests or debt at the end of period 2, such that she is subject to the period-by-period budget constraints

$$\begin{aligned} c_1 + s &= w_1 \\ c_2 &= w_2 + (1 + r)s, \end{aligned}$$

where s denotes the amount of savings. The consumer's labor income is w_t in each period and her preferences over consumption can be represented by the utility function

$$U(c_1, c_2) = \log(c_1) + \beta \log(c_2), \quad 0 < \beta < 1. \tag{1}$$

For the moment we abstract from the production side of the economy and simply assume that the consumer can borrow and lend consumption across periods at the given real interest rate, $r > 0$. We assume implicitly that the depreciation rate of capital is zero, $\delta = 0$.

(a) (20 Points) Write down the consumer's net present value budget constraint, and show that the optimal consumption in period 1 is given by

$$c_1 = \frac{1}{1 + \beta} \left(w_1 + \frac{w_2}{1 + r} \right).$$

State also the optimal savings.

Solution:

XX Allocation of points suggestions: present-value budget constraint and Lagrangean (10 Points), optimality conditions (5 Points), optimal consumption (3 Points), optimal savings (2 Points) XX Substituting out the savings, s, in the period-by-period budget constraint yields the net present value private budget constraint

$$c_1 + \frac{c_2}{1 + r} = w_1 + \frac{w_2}{1 + r}. \tag{2}$$

Maximizing $U(c_1, c_2)$ subject to the lifetime budget constraint in Equation (2) yields the first-order optimality conditions for consumption (let λ denote the Lagrange multiplier on the lifetime budget constraint)

$$\begin{aligned} 0 &= c_1^{-1} - \lambda \\ 0 &= \beta c_2^{-1} - \lambda/(1 + r). \end{aligned}$$

Combining the two yields the consumption Euler equation

$$c_2/c_1 = \beta(1 + r). \tag{3}$$

Combining Equations (3) and (2) yields

$$c_1 + \frac{\beta(1+r)}{1+r} c_1 = c_1(1+\beta) = w_1 + \frac{w_2}{1+r},$$

which can be reformulated as first period consumption

$$c_1 = \frac{1}{1+\beta}\left(w_1 + \frac{w_2}{1+r}\right).$$

Optimal savings will then be

$$\begin{aligned} s &= w_1 - c_1 \\ &= w_1\left(\frac{1+\beta}{1+\beta} - \frac{1}{1+\beta}\right) - \frac{1}{1+\beta}\frac{w_2}{1+r} \\ &= \frac{1}{1+\beta}\left(\beta w_1 - \frac{w_2}{1+r}\right). \end{aligned}$$

(b) (10 Points) In the above analysis you will have found that the optimal consumption growth over the life-cycle satisfies the Euler equation

$$\frac{c_2}{c_1} = \beta(1+r).$$

What is the elasticity of intertemporal substitution (EIS)

$$\text{EIS} = \frac{\partial \log(c_2/c_1)}{\partial \log(1+r)},$$

of this model specification then?

Solution:

Take logarithms on both sides of the Euler equation to yield

$$\log(c_2/c_1) = \log(\beta) + \log(1+r).$$

Thus the EIS is equal to 1.

(c) (10 Points) Compute the effect of an increase in the gross real interest rate $1+r$ (remember that this corresponds to an increase in the price of c_1 relative to c_2) on first-period consumption c_1. Is this the income, substitution, or wealth effect of the price change and how does your answer relate to the EIS derived in part (b)?

Solution:

The derivative of first-period consumption with respect to the gross interest rate yields (*XX 5 Points XX*)

$$\frac{\partial c_1}{\partial(1+r)} = -\frac{1}{1+\beta}\frac{w_2}{(1+r)^2} < 0.$$

This effect on first period consumption is a pure wealth effect (*XX 2 Points XX*), as with a EIS = 1 (or with logarithmic preferences) the substitution and income effect of a relative price change in consumption cancel exactly out (*XX correct argumentation related to part (b), 3 Points XX*).

We now turn from the representative consumer behavior to the economy as a whole. Suppose that this economy is populated by an infinite sequence of overlapping generations that live for two periods. Each generation is of size, L_t, where

$$L_{t+1} = (1+n)L_t,\ n > 0,\ L_0 > 0,$$

and an individual's old-age income is assumed to be zero,

$$w_2 = 0.$$

There is a production sector that combines aggregate physical capital, K_t, and labor, L_t, according to the technology

$$Y_t = F(K_t, Y_t L_t) = K_t^{\alpha}(A_t L_t)^{1-\alpha}, \quad 0 < \alpha < 1,$$

to produce output Y_t. Markets are competitive such that wage rate and the rental rate of capital are given by their marginal product

$$\begin{aligned} w_t &= (1-\alpha)A_t k_t^{\alpha}, \quad k_t \equiv K_t/(A_t L_t), \\ r_t &= \alpha k_t^{\alpha-1}, \end{aligned}$$

and $A_{t+1} = (1+g)A_t$, $g > 0$, $A_0 > 0$. Young agents save by buying unit claims to next period's capital stock, such that capital market clearing requires that the aggregate savings of the young, S_t, corresponds to the next period physical capital stock

$$S_t \equiv s_t L_t = K_{t+1}, \tag{4}$$

where s_t denotes the savings per capita of the current young.

(d) (10 Points) Compute the aggregate savings, S_t, in this economy and use the capital market clearing condition in Equation (4) to characterize the future capital stock K_{t+1} as a function of the current A_t, k_t and L_t. (Hint: if you were not able to solve for individual consumption and savings in part (a), you can assume that a constant fraction of the wage income is saved by each household,

$$s_t = \gamma w_t, \quad 0 < \gamma < 1,$$

to make further progress.)

Solution:

Aggregate savings are given by (only young agents earn a wage, so their lifetime income is simply w_t)

$$S_t = s_t L_t = \frac{\beta}{1+\beta} w_t L_t$$

substituting for the wage rate, the next period aggregate capital stock can be written as

$$K_{t+1} = \frac{\beta}{1+\beta}(1-\alpha)A_t k_t^{\alpha} L_t. \tag{5}$$

Note that $\gamma \equiv \beta/(1+\beta)$.

(e) (10 Points) Derive the law of motion for the capital stock per efficiency unit, k_{t+1} as a function of k_t, sketch it in a diagram with k_{t+1} on the vertical and k_t on the horizontal axis, and mark the stable steady state in the diagram (you do not have to compute the steady state).

Solution:

Multiply Equation (5) by $1/(A_t L_t)$ to yield

$$(1+g)(1+n)k_{t+1} = \frac{\beta}{1+\beta}(1-\alpha)k_t^{\alpha},$$

such that

$$k_{t+1} = \frac{\beta(1-\alpha)}{(1+\beta)(1+g)(1+n)}k_t^{\alpha}.$$

The future capital stock per efficiency unit, k_{t+1}, is a concave function in k_t with $k_{t+1}(0) = 0$. The stable steady state capital stock is characterized by the point of intersection between the $k_{t+1}(k_t)$ function and the 45-degree line where $k_{t+1} = k_t \equiv k^{\star} > 0$.

(f) (10 Points) Suppose the economy is in the stable steady state. Suddenly, in period t_0, due to a natural desaster half of the aggregate capital stock is destroyed. Sketch the dynamics of the capital stock per efficiency unit caused in response to this unexpected shock. Also, in a separate time diagram, sketch the dynamics of the logarithm of the wage rate over time. Be explicit in the diagrams whether a variable falls/increases by more or less than half on impact.

Solution:

In t_0 the capital stock per efficiency unit will jump down by half as half of the aggregate capital stock is destroyed. However, that triggers additional aggregate capital accumulation (due to the high interest rate) until the economy ends up with the same capital stock per efficiency unit as the economy converges back to the steady state.

The log-wage will fall on impact with the capital stock per efficiency unit (although by less than half due to the concavity in k_t). As the economy converges back to the steady state, wages will recover and end up on the same trajectory (remember that there is the trend growth of technology in the wage) as if the shock never had happened. The steady state wage with or without the shock is in both cases is

$$w_t = (1-\alpha)A_t(k^{\star})^{\alpha}.$$

Exercise C:
Long Question (60 Points)

A real business cycle model

Consider a representative household of a closed economy. The household has a planning horizon of two periods and is endowed with the following preferences over consumption, c,

$$U = u(c_1) + \beta \mathrm{E} u(c_2(s_2)),$$

with the following marginal utility

$$u'(c) = c^{-\gamma},\ \gamma \geq 1.$$

The variable s_2 denotes the state of the economy in the second period which follows the stochastic process

$$s_2 = \begin{cases} s_G, \text{ with prob. } p \\ s_B, \text{ with prob. } 1-p, \end{cases}$$

and the household conditions the consumption, $c_2(s_2)$, in the second period on the state, s_2. Assume the household's labor supply is exogenous and always equal to 1.

Labor market assumptions:

Assume that in each period and in each state of the economy, s_t, there is a linear (in labor n_t) production technology of the form

$$y_t(s_t) = A_t(s_t) n_t(s_t),$$

and the labor market is assumed to be competitive. Assume the labor productivity in the first period is given by $A_1 = A$, and the labor productivity is higher in the good state of the second period,

$$A_2(s_G) = A + A(1-p)\epsilon > A_2(s_B) = A - Ap\epsilon, \quad \epsilon > 0, A > 0, 0 < p < 1,$$

than in the bad state of the second period. The wages are denoted as w_1, $w_2(s_G)$, and $w_2(s_B)$.

Asset market assumptions:

Assume the household does have access to a risk-free asset, a_2, and the associated interest rate is denoted as r_2.

(a) (5 Points) Find the equilibrium wages, w_1, $w_2(s_G)$, and $w_2(s_B)$, and show that the expected wages in the second period is the same as wage in in the first period.

Solution:

Since labor markets are competitive, we should have wages equal to productivity as follows:

$$\begin{aligned} w_1 &= A, \\ w_2(s_G) &= A_2(s_G), \\ w_2(s_B) &= A_2(s_B). \end{aligned}$$

***XX Allocation of points**:*
3 points for get the wages correct ; 2 points for the proof is correct

(b) (5 Points) Write down the state-by-state budget constraints for the household.

Solution:

the state-by-state budget constraints for the household are:

$$\begin{aligned} c_1 + a_2 &= w_1, \\ c_2(s_2) &= w_2(s_2) + (1 + r_2)a_2, \forall s_2 \in S \equiv \{s_G, s_B\}. \end{aligned}$$

***XX Allocation of points**:*
1 point for c_1; 2 points for each of the other quantities

(c) (10 Points) Let $(\lambda_1, \lambda_2(s_G), \lambda_2(s_B))$ denote the Lagrange multipliers of the state-by-state budget constraints. State the representative agent's Lagrangian. (Note that the expected utility for the second period is the summation of utility across good and bad states, weighted by probability, i.e., $\mathrm{E}u(c_2(s_2)) = pu(c_2(s_G)) + (1 - p)u(c_2(s_B))$.)

Solution:

The state-by-state budget constraints are:

$$\begin{aligned} c_1 + a_2 &= w_1 \times 1, \\ c_2(s_2) &= w(s_2) \times 1 + (1 + r_2)a_2, \forall s_2 \in S \equiv \{s_G, s_B\}. \end{aligned}$$

The Lagrangian can be written in the state-ordered form as

$$\begin{aligned} \mathcal{L} = u(c_1) &+ \lambda_1 [w_1 - a_2 - c_1] \\ &+ \beta p [u(c_2(s_G))] + \lambda_2(s_G) [w(s_G) + (1 + r_2)a_2 - c_2(s_G))] \\ &+ \beta(1 - p) [u(c_2(s_B))] + \lambda_2(s_B) [w(s_B) + (1 + r_2)a_2 - c_2(s_B))] . \end{aligned}$$

XX Allocation of points:
10 points for the Lagrangian: if correct or make sense (could be in other formulations); deduct 3 points if one of the three blocks wrong.

(d) (10 Points) Derive the stochastic consumption Euler equation (only involves $c_1, c_2(s_2), \beta$ and r_2 and NO multipliers).

Solution:

The optimality conditions with respect to the choices are:

$$0 = \frac{\partial \mathcal{L}}{\partial c_1} = u'(c_1) - \lambda_1 \tag{6}$$

$$0 = \frac{\partial \mathcal{L}}{\partial c_2(s_G)} = \beta p u'(c_2(s_G)) - \lambda_2(s_G) \tag{7}$$

$$0 = \frac{\partial \mathcal{L}}{\partial c_2(s_B)} = \beta(1-p)u'(c_2(s_B)) - \lambda_2(s_B) \tag{8}$$

$$0 = \frac{\partial \mathcal{L}}{\partial a_2} = -\lambda_1 + [\lambda_2(s_G) + \lambda_2(s_B)](1+r_2). \tag{9}$$

The stochastic consumption Euler equation is given by:

$$u'(c_1) = \beta \mathrm{E}\left[u'(c_2(s_2))\right](1+r_2). \tag{10}$$

XX Allocation of points:
10 points if the final formula correct; otherwise, only 1 point for each of the focs.

(e) (10 Points) For (e), (f) and (g), assume that the asset a_2 is available in zero supply. What is the household's optimal choice of a_2 in the equilibrium? What are the household's optimal choices of consumption? Can the household fully smooth consumption? i.e., are c_1, $c_2(s_G)$ and $c_2(s_B)$ equal?

Solution:

In equilibrium $a_2 = 0$ since we assume this is a representative household and zero net asset supply. The state-by-state budget constraints imply the following consumption levels

$$c_1 = w_1$$
$$c_2(s_2) = w(s_2),\ \forall s_2 \in S,$$

or,

$$\begin{aligned} c_1 &= w_1 = A, \\ c_2(s_G) &= w_2(s_G) = A_2(s_G) > A, \\ c_2(s_B) &= w_2(s_B) = A_2(s_B) < A. \end{aligned}$$

so consumption is not fully smoothed.

XX Allocation of points*:*
4 points to get the savings correct. 2 points for each of the consumption quantities.

(f) (10 Points) Assume now (for this part only) that $A_1 = 2$ in period 1, $\epsilon = 1$, and $\beta = 1/2$, $\gamma = 1$ and $p = 1/2$. What is the gross interest rate $1 + r$ in equilibrium?

Solution:

The equilibrium gross interest rate is given by

$$1 + r = \frac{1/2}{1/2\,[1/2 \times 1/3 + 1/2 \times 1]} = \frac{1}{1/6 + 3/6} = 3/2.$$

(g) (10 Points) Is the equilibrium interest rate r_2 higher or lower than $r_{RN} \equiv \frac{1}{\beta} - 1$? Why? (Hint: do it step by step: (1) use the budget constraint to link consumption and wages; (2) use the Euler equation and the result, $u'(w_1) \leq \mathrm{E}[u'(w(s_2))]$, which comes from the Jensen's inequality.)

Solution:

In the stochastic economy with $\epsilon > 0$, Jensen's inequality implies

$$(1 + r_2)\beta = \frac{u'(c_1)}{\mathrm{E}\left[u'(c(s_2))\right]} = \frac{u'(w_1)}{\mathrm{E}\left[u'(w(s_2))\right]} = \frac{u'(\mathrm{E}\left[w(s_2)\right])}{\mathrm{E}\left[u'(w(s_2))\right]} < 1,$$

such that the interest rate in this economy is smaller than the risk neutral interest rate, $r_{RN} \equiv \frac{1}{\beta} - 1$.

XX Allocation of points*:*

(1) 5 points: get the budget constraint and use the Euler equation correctly

(2) 5 points: use the inequality and compare the two interest rates correctly