ECON4510 – Finance Theory
Lecture 1

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Administrative

- Please check course web site often (messages, exercises, etc.):
- http://www.uio.no/studier/emner/sv/oekonomi/ECON4510/h13
- Will also use Fronter for seminars and exercises
- 13 lectures of 2 x 45 minutes, once weekly, Mondays 10:15–12:00
- No lecture on 30 Sep
- 6 seminars of 2 x 45 minutes (irregular dates 27 Aug →, see plan)
- Grade based only on final exam 28 Nov; 14:30–17:30; closed book
- Essential to work with (more than one) seminar assignments to prepare for exam
- There will also be other exercises, with suggested answers
- Lecture notes (like these) are on web site the Friday before each lecture
- Lectures in English, but Norwegian translation when asked for
- You may ask questions in Norwegian during lectures, will be translated, then answered

Overview

- Main topic: What are the values of various assets?
- Both financial and real assets: Securities (shares of stock, bonds, options, etc.), investment projects, property
- Central feature of theories: Uncertainty about future income streams connected to the assets, or their values in the future
- Equilibrium models: Supply and demand determine values
- Applications in firms and business:
  - Determine values for trading assets
  - Decision tool for investment projects
  - Answer questions like: Should firms diversify?
- Applications in government:
  - Privatization (or acquiring assets)
  - Decision tool for investment projects
  - Regulation of markets
  - Taxation of firms

Overview, contd.

- You will *not* learn how to make money in the markets
- In fact, you will learn why that is very difficult
- You will learn basic theory about what determines (and what does *not* influence) security equilibrium prices
- You will also learn about the role of financial markets in the economy
  - Desynchronize (separate consumption from income) in time
  - Desynchronize between outcomes (states of nature)
  - Welfare economics under uncertainty
- This course does not cover control of firms, or conflicts due to asymmetries of information between management, shareholders, and lenders. Those topics: ECON4245 Corporate Governance

Required background and overlap

- This course builds on mathematics at the level of ECON3120/4120; those who do not have it, should take that course in parallel
- This course builds on statistics at the level of ECON2130
- More math, such as ECON4140, and statistics, such as ECON4130, is an advantage
- This course overlaps with ECON3200/4200 on the topic of decisions under uncertainty, “expected utility”, but full credit is given anyhow
- Will refer to textbooks by Sydsæter et al.:

MA1: Sydsæter, *Matematisk Analyse bd. 1,* seventh or eighth edition, 2000 or 2010

MA2: Sydsæter, Seierstad and Strøm, *Matematisk Analyse bd. 2,* fourth edition, 2002

EMEA: Sydsæter and Hammond, *Elementary Mathematics for Economic Analysis,* third or fourth edition, 2008 or 2012

FMEA: Sydsæter, Hammond, Seierstad and Strøm, *Further Mathematics for Economic Analysis,* second edition, 2008

Equilibrium models vs. arbitrage pricing (D&D ch. 2)

Two very different theoretical starting points

Equilibrium model:

- Determine prices by supply and demand
- The equilibrium prices depend on everything in the model, such as the preferences of the agents, their endowments W_0^h, perhaps some exogenous variables (typically: the risk free interest rate)
- Complicated, but some of the results fairly simple

Arbitrage pricing:

- Determines prices by correspondence with other existing assets
- Argument: Since this asset gives the same future cash flow as some other (set of) asset(s), it must have the same value today
- If not, there would be opportunities of arbitrage, making money by buying and selling at observed market prices
- Conceptual problem: If we find how a price must relate to some other price(s), what if these change? (Equilibrium?)

Equilibrium models vs. arbitrage pricing, contd.

In this course: Will first concentrate on equilibrium models, later (D&D ch. 10, and option theory, Hull) on arbitrage models

Surprisingly, the practical difference between the two types of models does not need to be big

Arbitrage pricing particularly useful for options and similar securities, whose prices obviously depend on prices of other securities (typically stocks)

Preview of practical results (D&D sect. 2.2)

- Practical focus: $V(\tilde{X})$, value today of future cash flow $\tilde{X}$
- Background: Why not just take present value of $E(\tilde{X})$?
- One particular principle will be important: Value additivity
- Justify later that $V(\tilde{X}_1 + \tilde{X}_2) = V(\tilde{X}_1) + V(\tilde{X}_2)$
- With this in mind, what does $V()$ function look like?
  - Alt. 1: Risk-adjusted discount rate,

    $$\frac{E(\tilde{X})}{1 + r_f + \pi},$$

    where π is risk premium added to risk-free interest rate r_f
  - Alt. 2: Present value (PV) of risk-adjusted expectation,

    $$\frac{E(\tilde{X}) - \Pi}{1 + r_f},$$

    where r_f is used to find PV, but a deduction Π is made in $E(\tilde{X})$

Preview of practical results, contd.

- What does $V()$ function look like, contd.
  - Alt. 3: Expected present value based on adjustment in probability distribution,

    $$\frac{\hat{E}\tilde{X}}{1+r_f},$$

    where $\hat{E}$ represents those adjusted probabilities
  - Alt. 4: Pricing based on state-contingent outcomes, $X(\theta)$,

    $$\Sigma_\theta \;\; q(\theta)X(\theta),$$

    where $q(\theta)$ is value of claim to one krone in state θ

Choice under uncertainty

- In order to construct theoretical model of asset markets: Need theory of people's behavior in these markets
- Choice "under uncertainty," i.e., between risky alternatives
- Example:
  - May buy government bonds and earn interest at a known rate
  - May alternatively buy shares in the stock market with risky returns
  - E.g., invest everything in one company, such as Norsk Hydro
  - One certain, one uncertain alternative
- In reality many uncertain alternatives: Shares in different companies
- May diversify: Invest some money in one company, some in another
- May also invest outside of asset markets, "real investment" projects
- Outcome one year into the future of each choice is uncertain
- Assume the outcome in each alternative can be described by a probability distribution
- Exist also theories of choice under "total uncertainty" without probabilities, but much more difficult

Choice under uncertainty, contd.

- Choice between probability distributions of consumption in future periods
- Simplification in finance: Only one good, money (but theory in chapters 1 and 8 in D&D can deal with vectors of different goods)
- To begin with: Uncertainty in *one period* only
- Choices are made now (often called period zero), with uncertainty about what will happen next (period one)
- Only one future period: Consumption = wealth in that period
- Will return later to situation with more than one future date

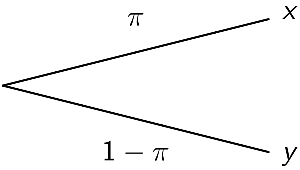
Choice under uncertainty, contd.

- Each choice alternative (e.g., invest 50% in bank account and 50% in a particular company's shares) gives one probability distribution of outcomes in period one
- *All consequences* and the total situation of the decision maker should be taken into consideration when choices are described; for instance:
  - Choose between (a) keeping \$10 and (b) spending it on a lottery ticket with 1 per cent probability of winning \$1000 and 99 per cent of loss
  - This is *different* from the problem, when \$10000 is added, of choosing between \$10010 on one hand and on the other a 1 per cent probability of \$11000 and 99 per cent probability of \$10000
  - One will often find that this addition of \$10000 (across both alternative actions) increases the willingness to take on risk, so that more people would choose the more risky alternative in the case with higher total consumption
  - We will, during the course, make more precise what is meant by such an income effect on the choice

von Neumann and Morgenstern's theory

"Expected utility"

Objects of choice called *lotteries.* Simplification: Each has only two possible, mutually exclusive outcomes. Notation: $L(x, y, \pi)$ means:



(The $L()$ notation means: The first two arguments are outcomes. Then comes the probability (here: $\pi \in [0, 1]$) of the outcome mentioned first (here: x).)

von Neumann and Morgenstern's theory, contd.

Axiom C.2 (D&D, p. 45) says that an individual is able to compare and choose between such stochastic variables, and that preferences are transitive.

Axiom C.3 says that preferences are continuous.

Assumptions like C.2 and C.3 are known from standard consumer theory.

Axiom C.1 says that only the probability distribution matters.

Axioms C.4–C.7 specific to preferences over *lotteries.*

The theory assumes axioms C.1–C.7 hold for the preferences of one individual. Using the theory, we usually assume it holds for all individuals, but their preferences may vary within the restrictions given by the theory.

von Neumann and Morgenstern's theory, contd.

Axiom C.4 *Independence:* Let x, y and z be outcomes of lotteries. In fact, x, y, and/or z could be new lotteries. Assume $y \succsim z$, "y is weakly preferred to z." Then

$$L(x, y, \pi) \succsim L(x, z, \pi).$$

Axiom C.5 Among all lotteries (and outcomes), there exists one best lottery, b, and one worst, w, with $b \succ w$, "b is strictly preferred to w."

von Neumann and Morgenstern's theory, contd.

Axiom C.6 If $x \succ y \succ z$, then there exists a unique π such that

$$y \sim L(x, z, \pi).$$

(Not obvious. What about life and death?)

Axiom C.7 Assume $x \succ y$. Then

$$L(x, y, \pi_1) \succ L(x, y, \pi_2) \Leftrightarrow \pi_1 > \pi_2,$$

(Actually: None of axioms are obvious.)

Derivation of theorem of expected utility

With reference to b and w, for all lotteries and outcomes z, define a function $\pi()$ such that

$$z \sim L(b, w, \pi(z)).$$

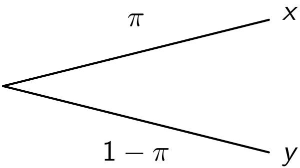
This probability exists for all z by axiom C.6. By axiom C.7 it is unique and can be used to rank outcomes, since $\pi(x) > \pi(y) \Rightarrow x \succ y$. Thus $\pi()$ is a kind of utility function.

Will prove it has the expected utility property: The utility of a lottery is the expected utility from its outcomes.

Digression: A *utility function* for a person assigns a real number to any object of choice, such that a higher number is given to a preferred object, and equal numbers are given when the person is indifferent between the objects. If x and y are money outcomes or otherwise quantities of a (scalar) good, and there is no satiation, then π is an increasing function.

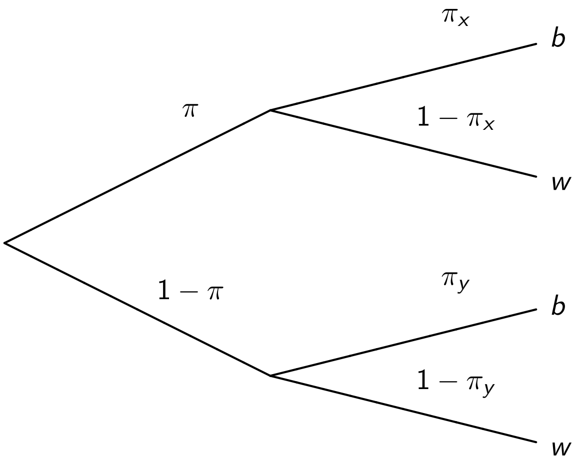
The expected utility property

Consider a lottery $L(x, y, \pi)$, which means:

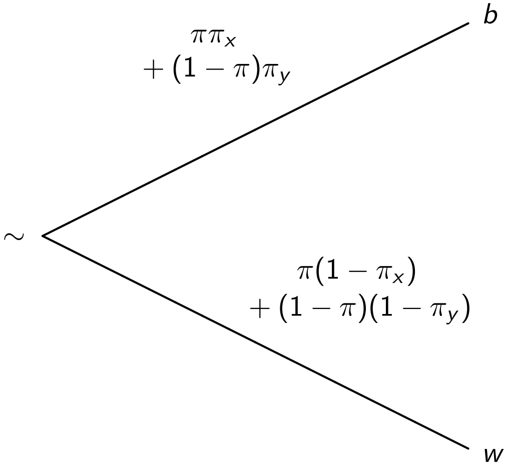


Indifference between $L(x, y, \pi)$ and compound lottery

When $x \sim L(b, w, \pi_x)$ and $y \sim L(b, w, \pi_y)$, then there will be indifference between $L(x, y, \pi)$ and the lotteries on this and the following page:



Simplification of the compound lottery



The expected utility property, conclude

So that:

$$L(x, y, \pi) \sim L(b, w, \pi\pi_x + (1 - \pi)\pi_y).$$

Thus the "utility" of $L(x, y, \pi)$ is $\pi\pi_x + (1 - \pi)\pi_y$.

The expected utility property, summarize

"Utility" of a lottery was defined by finding a lottery with outcomes b, w which is seen as equally attractive as the first one. The utility number is the probability of b in that second lottery. The utility of $L(x, y, \pi)$ was found to be $\pi\pi_x + (1 - \pi)\pi_y$.

This turns out to have exactly the promised form: It is the expectation of a random variable which takes the value π_x with probability π and π_y with probability $1 - \pi$. These two outcomes, π_x and π_y are exactly the utility numbers for x and y, respectively.

The utility expression $\pi\pi_1 + (1 - \pi)\pi_2$ can be interpreted as *expected utility.*

Notation: Usually the letter U is chosen for the utility function instead of π, and expected utility is written $E[U(\tilde{X})]$.

The expected utility property, extend

Possible to extend to ordering of lotteries of more than two outcomes,

$$E[U(\tilde{X})] = \sum_{s=1}^{S} \pi_s U(x_s),$$

even to a continuous probability distribution,

$$E[U(\tilde{X})] = \int_{-\infty}^{\infty} U(x) f(x) dx.$$

Will not look at this more formally.

Criticism of vN-M expected utility

- Some experiments indicate that many people's behavior in some situations contradicts expected utility maximization.
- Exist alternative theories, in particular generalizations (alternative theories in which expected utility appears as one special case).
- Nevertheless much used in theoretical work on decisions under uncertainty.

Example of when vN-M may not work

- Suppose every consumption level below 5 is very bad.
- Suppose, e.g., that $U(4) = -10, U(6) = 1, U(8) = 4, U(10) = 5$.
- Then $E[U(L(4, 10, 0.1))] = 0.1 \cdot (-10) + 0.9 \cdot 5 = 3.5$, while $E[U(L(6, 8, 0.1))] = 0.1 \cdot 1 + 0.9 \cdot 4 = 4.6$.
- But even with the huge drop in U level when consumption drops below 5, one will prefer the first of these two alternatives (the lottery $L(4, 10, \pi)$) to the other ($L(6, 8, \pi)$) as soon as π drops below $1/12$.
- If instead one outcome is so bad that someone will avoid it *any* cost, even when its probability is very low, then that person's behavior contradicts the vN-M theory.
- In particular, axiom C.6 is contradicted.

Allais paradox

Behavior at odds with vN-M theory, observed by French economist Maurice Allais. Consider the following lotteries:

- $L^3 = L(10000, 0, 1)$
- $L^4 = L(15000, 0, 0.9)$
- $L^1 = L(10000, 0, 0.1) = L(L^3, 0, 0.1)$
- $L^2 = L(15000, 0, 0.09) = L(L^4, 0, 0.1)$

People asked to rank L^1 versus L^2 often choose $L^2 \succ L^1$. (Probability of winning is just slightly less, while prize is 50 percent bigger.)

But when the same people are asked to rank L^3 versus L^4, they often choose $L^3 \succ L^4$. (With strong enough risk aversion, the drop in probability from 1 to 0.9 is enough to outweigh the gain in the prize.) Is this consistent with the vN-M axioms?

Using C4, if $L^3 \succsim L^4$, then . . . .

Uniqueness of U function

Given a vN-M preference ordering of one individual, have now shown we can find a U function such that

$$\tilde{X} \succ \tilde{Y} \text{ if and only if } E[U(\tilde{X})] > E[U(\tilde{Y})].$$

Considering one individual, we ask: Is U unique? No, depends on b and w, but there is no reason why preferences between $\tilde{X}$ and $\tilde{Y}$ should depend on b or w.

Define an increasing linear transformation of U,

$$V(x) \equiv c_1 U(x) + c_0,$$

where $c_1 > 0$ and c_0 are constants. This represents the preferences of the same individual equally well since

$$E[V(\tilde{X})] = c_1 E[U(\tilde{X})] + c_0$$

for all $\tilde{X}$, so that a higher $E[U(\tilde{X})]$ gives a higher $E[V(\tilde{X})]$, and vice versa.

Uniqueness of U function, contd.

But *not* possible to do similar replacement of U with any *non*-linear transformation of U (as opposed to ordinal utility functions for usual commodities).

For instance, $E\{\ln[U(\tilde{X})]\}$ does not necessarily increase when $E[U(\tilde{X})]$ increases. So $\ln[U()]$ cannot be used to represent the same preferences as $U()$.

Risk aversion

For those preference orderings which (i.e., for those individuals who) satisfy the seven axioms, define *risk aversion*.

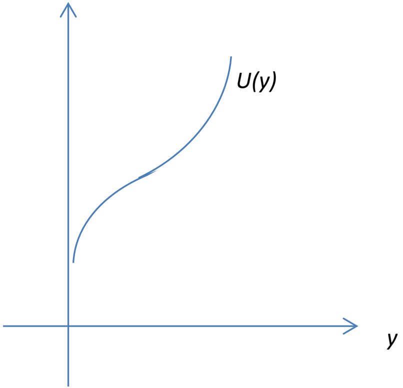
Compare a lottery $\tilde{Y} = L(a, b, \pi)$ (where a, b are fixed monetary outcomes) with receiving $E(\tilde{Y}) = \pi a + (1 - \pi)b$ for sure. Whether the lottery, $\tilde{Y}$, or its expectation, $E(\tilde{Y})$, is preferred, depends on the curvature of U:

- If U is linear, then $U[E(\tilde{Y})] = E[U(\tilde{Y})]$, and one is indifferent between lottery and its expectation. One is called *risk neutral*.
- If U is concave, then $U[E(\tilde{Y})] \geq E[U(\tilde{Y})]$, and one prefers the expectation. One is called *risk averse*.
- If U is convex, then $U[E(\tilde{Y})] \leq E[U(\tilde{Y})]$, and one prefers the lottery. One is called *risk attracted*.
- (A fourth category? See next page.)

Risk aversion and concavity

The inequalities follow from *Jensen's inequality* (see *MA2,* sect. 4.5, *FMEA,* sect. 2.4, or D&D, p.63). If U is *strictly* concave or convex, the inequalities are strict, except if $\tilde{Y}$ is constant with probability one.

Quite possible that many have U functions which are neither everywhere linear, everywhere concave, nor everywhere convex. Then those people do not fall into any of the three categories.



From now on: Assume risk aversion

Even though risk aversion does *not* follow from the seven axioms, will assume, for the remainder of the course, that people are risk averse

- Most common behavior in economic transactions.
- Explains the existence of insurance markets.
- But what about money games? Expected net result always negative, so a risk-averse should not participate. Cannot be explained by theories taught in this course.
- Some of our theories will collapse if someone is risk neutral or risk attracted. Those will take all risk in equilibrium. Does not happen.

Arrow-Pratt measures of risk aversion

How to measure risk aversion?

- Natural candidate: $-U''(y)$. (Why minus sign?)
- Varies with the argument, e.g., high y may give lower $-U''(y)$.
- Is $U()$ twice differentiable? Assume yes.
- But: The magnitude $-U''(y)$ is not preserved if $c_1 U() + c_0$ replaces $U()$.
- Use instead:
  - $-U''(y)/U'(y)$ measures *absolute risk aversion.*
  - $-U''(y)y/U'(y)$ measures *relative risk aversion.*
- In general, these also vary with the argument, y.

Risk premium

- Will introduce the concept *risk premium*, related to expected utility. This concerns a situation in which we have specified the complete, uncertain consumption (or income or wealth) which is the argument of the (expected) utility function.
- (Later we will consider the pricing of securities in a stock market. The required expected rate of return on a security will have a term which reflects the security's risk in relation to the market. This term could also be called a risk premium, but this is a very different concept, and you will see why.)
- Will also say more about the two measures of risk aversion.
- Will show on next pages: For small risks, $R_A(y) \equiv -U''(y)/U'(y)$ meaures how much compensation a person demands for taking the risk. Called the Arrow-Pratt measure of absolute risk aversion.
- $R_R(y) \equiv -U''(y) \cdot y/U'(y)$ is called the Arrow-Pratt measure of relative risk aversion.

Absolute risk aversion, relation to risk premium

- Consider the following case (somewhat more general than D&D, sect. 4.3.1):
  - The wealth Y is non-stochastic.
  - A lottery $\tilde{Z}$ has expectation $E(\tilde{Z}) = 0$.
- For a person with utility function $U()$ and inital wealth Y, define the *risk premium* Π associated with the lottery $\tilde{Z}$ by

$$E[U(Y + \tilde{Z})] = U(Y - \Pi).$$

- Will show the relation between Π and absolute risk aversion.

Risk premium is proportional to risk aversion

(The result holds approximately, for small lotteries.)

Risk premium, Π, is defined by

$$E[U(Y+\tilde{Z})] \equiv U(Y-\Pi).$$

Take quadratic approximations (second-order Taylor series) (*MA1,* sect. 7.4–5, *EMEA,* sect. 7.4–5) LHS:

$$U(Y+z) \approx U(Y) + zU'(Y) + \frac{1}{2}z^2U''(Y)$$

which implies

$$E[U(Y+\tilde{Z}]) \approx U(Y) + \frac{1}{2}E(\tilde{Z}^2)U''(Y).$$

RHS:

$$U(Y-\Pi) \approx U(Y) - \Pi U'(Y) + \frac{1}{2}\Pi^2U''(Y).$$

Risk premium is proportional to risk aversion

Use the notation $\sigma_z^2 \equiv \text{var}(\tilde{Z}) = E(\tilde{Z}^2) - [E(\tilde{Z})]^2 = E(\tilde{Z}^2)$ since $E(\tilde{Z}) = 0$. Since Π is small, Π^2 is very small. Thus the last term on the RHS is very small, and we will neglect it. Then we are left with:

$$\frac{1}{2}\sigma_z^2 U''(Y) \approx -\Pi U'(Y)$$

which implies the promised result:

$$\Pi \approx -\frac{U''(Y)}{U'(Y)} \cdot \frac{1}{2}\sigma_z^2.$$

The U function: Forms which are often used

- Some theoretical results can be derived without specifying form of U.
- Other results hold for specific classes of U functions.
- *Constant absolute risk aversion (CARA)* holds for $U(y) \equiv -e^{-ay}$, with $R_A(y) = a$.
- *Constant relative risk aversion (CRRA)* holds for $U(y) \equiv \frac{1}{1-g} y^{1-g}$, with $R_R(y) = g$.
- (Exercise: Verify these two claims. (a, g are constants.) Determine what are the permissible ranges for y, a and g, given that functions should be well defined, increasing, and concave.)
- Essentially, these are the only functions with CARA and CRRA, respectively, apart from CRRA with $R_R(y) = 1$.
- (Without affecting preferences: Any constant can be added to the functions; any constant > 0 can be multiplied with them.)

The U function: Forms which are often used, contd.

- $R_R(y) \equiv 1$ is obtained with $U(y) \equiv \ln(y)$.
- Another much used form: $U(y) = -ay^2 + by + c$, quadratic utility. Easy for calculations, U' linear.
- (What are permissible ranges, given that U should be concave? Hint: There is a minus sign in front of a.)
- Quadratic U has increasing $R_A(y)$ (Verify!), perhaps less reasonable.
- (What happens for this U function when $y > b/2a$? Is this reasonable?)

Stochastic dominance

- Two criteria for making decisions without knowing shape of $U()$.
- May be important for delegation, for research, for prediction: Situations in which you are not able to point out exactly which U function is the right one to use.
- These two criteria (see below) work only for some types of comparisons. For other comparisons, these decision criteria are inconclusive.
- When you have many (more than two) alternatives, it will often turn out that neither of the two dominance criteria give you an answer to which alternative is the best. But one of them (or both) can nevertheless be useful for narrowing down choices by excluding dominated alternatives.

First-order and second-order stochastic dominance

A random variable $\tilde{X}_A$ *first-order stochastically dominates* another random variable $\tilde{X}_B$ if every vN-M expected utility maximizer prefers $\tilde{X}_A$ to $\tilde{X}_B$.

A random variable $\tilde{X}_A$ *second-order stochastically dominates* another random variable $\tilde{X}_B$ if every *risk-averse* vN-M expected utility maximizer prefers $\tilde{X}_A$ to $\tilde{X}_B$.

When comparing two alternatives, let the cumulative distribution functions be $F_A(x) \equiv \Pr(\tilde{X}_A \leq x)$ and $F_B(x) \equiv \Pr(\tilde{X}_B \leq x)$.

First-order stochastic dominance, FSD

Possible to show that "$\tilde{X}_A \succ \tilde{X}_B$ by all" is equivalent to the following, which is one possible definition of first-order s.d.:

$$F_A(w) \leq F_B(w) \text{ for all } w,$$

and

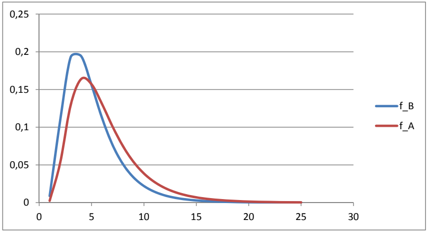
$$F_A(w_i) < F_B(w_i) \text{ for some } w_i.$$

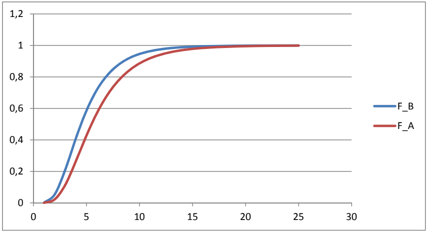
For any level of wealth w, the probability that $\tilde{X}_A$ ends up below that level is less than the probability that $\tilde{X}_B$ ends up below it.

First-order stochastic dominance, illustrated

Left diagram shows density functions of two alternatives. Red curve is more attractive (for what kind of persons?) since more probability mass is moved to the right.

Right diagram shows corresponding cumulative distribution functions. Red curve shows everywhere a lower probability of getting a lower (less attractive) outcome.





Second-order stochastic dominance, SSD

Possible to show that "$\tilde{X}_A \succ \tilde{X}_B$ by all risk averters" is equivalent to the following, which is one possible definition of second-order s.d.:

$$\int_{-\infty}^{w_i} F_A(w)dw \leq \int_{-\infty}^{w_i} F_B(w)dw \text{ for all } w_i,$$

and

$$F_A(w_i) \neq F_B(w_i) \text{ for some } w_i.$$

One distribution is more dispersed ("more uncertain") than the other. If we restrict attention to variables $\tilde{X}_A$ and $\tilde{X}_B$ with the same expected value, Theorem 4.4 in D&D states that SSD is equivalent to: $\tilde{X}_B$ can be written as $\tilde{X}_A + \tilde{z}$, where the difference $\tilde{z}$ is some random noise.

Second-order stochastic dominance, illustrated

Left diagram shows density functions of two alternatives. Red curve is more attractive (for what kind of persons?) since the probability mass is more concentrated.

Right diagram shows corresponding cumulative distribution functions. Red curve shows for low *w* values a lower probability of getting a lower (less attractive) outcome, but this is reversed for higher *w* values.