ECON4510 Finance theory — Diderik Lund, 27 August 2013

Seminar exercises 2–6, fall 2013

Instructions:

For all students: Please try to solve these exercises before the seminars. Solving problems is by far the best way to learn the material and prepare for the exam. It is generally a better idea to work a little with all questions than to work a lot with only a few of them.

For each seminar meeting, we ask for volunteers to prepare suggested solutions to that meeting's exercises. This has two purposes: Those who prepare something, will learn from it. Also, the others will learn, both from correct and mistaken parts of the suggestions. The organization of this, including distribution of the suggested solutions, will be discussed in the first seminar.

Complete, correct solutions to all problems will *not* be made available (except if/when the volunteers' contributions can be regarded as such). You will have to take part in the seminars in order to get all the information, corrections, discussions, modifications, etc. However, if there are some parts which we do not get through because of limited time, solutions to those remaining parts (or extensive comments to the volunteers' suggested solutions) will be made available.

In the texts of the exercises, the notation is sometimes inconsistent. For instance, the risk free interest rate can be r_f in one exercise, but R_f in another. In some cases there is no indication of which variables are stochastic, i.e., the risky rates of return are written as r_j, not $\tilde{r}_j$. This happens because the problems are collected from various sources, and you just have to get used to this. But within each problem, there should be consistency, and the notation should be explained.

The exercises for seminar no. 1 are available in a separate file.

As of today, 27 August 2013, this file is incomplete, because exercise no. 1 for the last seminar is missing.

Exercises for seminar no. 2, 10 September

(1)

(Part of exam question for ECON460, fall 2002, slightly rewritten.)

In this exercise you are asked to sketch two different opportunity sets in the same diagram, and then to consider what can be said about what choices an investor would make.

Consider a pair of risky securities, 1 and 2, with rates of return $r_j, j = 1, 2$, with the properties $\mu_1 = E(r_1) = 0.16$, $\sigma_1^2 = \text{var}(r_1) = 0.2^2 = 0.04$, $\mu_2 = E(r_2) = 0.08$, and $\sigma_2^2 = \text{var}(r_2) = 0.3^2 = 0.09$.

Two additional pieces of information are needed in order to define the opportunity set of portfolios from the two securities: Short selling may or may not be allowed (see below). With regard to the covariance, you are asked to consider two cases:

Case (i): The covariance between the rates of return is $\sigma_{12} = \text{cov}(r_1, r_2) = 0.045$.

Case (ii): The covariance between the rates of return is $\sigma_{12} = \text{cov}(r_1, r_2) = 0.015$.

1(a)

For case (i): Determine the composition of that portfolio of the two securities which has the minimum variance, and then show that this variance is $\sigma_{mv}^2 = 0.63/16 \approx 0.198^2$. Do likewise for case (ii), with the resulting $\sigma_{mv}^2 = 0.54/16 \approx 0.184^2$.

1(b)

Sketch the two hyperbolae in the same (σ, μ) diagram. Remember that each hyperbola is symmetric around a horizontal line at the μ level which gives the minimum variance (and minimum standard deviation).

1(c)

What are the two opportunity sets (for cases (i) and (ii), resp.) if short selling is allowed? If you only know that an investor is risk averse with mean-variance preferences, can you determine whether the investor would prefer to have the opportunity set given by case (i) or that given by case (ii)?

1(d)

What are the two opportunity sets if short selling is *not* allowed? If you only know that an investor is risk averse with mean-variance preferences, can you determine whether the investor would prefer to have the opportunity set given by case (i) or that given by case (ii)?

(2)

The lecture notes describe short selling of a risky security. Discuss whether this description is relevant (in practice, not only in theory) for two other forms of investment: (i) A risk free security. (ii) A risky real investment project.

(3)

The lecture notes consider situations with one or more risky assets, but there is never more than one risk free asset. Explain why a situation with many different risk free assets is not an interesting situation to consider.

(4)

In the lecture notes discussing the opportunity sets with more than one risky asset, it was implicitly assumed that the risky assets have different expected rates of return. What can you say about the opportunity set in a situation with only two risky assets when these have the same expected rate of return?

(5)

In the lecture introducing the CAPM, it was claimed that the frontier portfolio set with n risky assets, solving

$$\min_{w_1,\ldots,w_n} \sigma_p \text{ given } \mu_p$$

is a hyperbola, but there was no proof of this. For the subsequent derivation of the CAPM, however, there was no use for a formula describing the graph of the frontier portfolio set.

5(a)

Explain why the derivation must be based on an assumption (or even better, a proof) that the graph of $\sigma(\mu)$ (the frontier portfolio set transposed with μ as the argument of the function) is convex.

5(b)

Explain why we can be sure that the graph of $\sigma(\mu)$ is indeed convex. (Hint: Assume the opposite, that the curve is concave for some segment, say between μ_1 and μ_2. Why is that at odds with what you know about combining two risky assets?)

Exercises for seminar no. 3, 17 September

(1)

(Exam Spring 2008, problem 1, slightly rewritten.)

Consider a limited-liability company with shares traded in the stock market. Assume there are no taxes and that the company does not need to borrow. The company is considering to undertake its first real investment project.

It must make a choice in period zero of the output quantity to produce, $Q \geq 0$, which will require a quantity $C(Q)$ of an input factor. There are no other costs. $C(Q)$ is an increasing and convex function. Both quantities, Q and $C(Q)$, will be known with full certainty as soon as the decision on Q has been made, and there is no way to change these later. The output will be ready for sale in period one at an output price $\tilde{P}$, which is uncertain as seen from period zero.

Two alternative cases for payment for the input factor are considered in parts 1(a) and 1(b) below.

1(a)

Assume that the input factor must be paid for in period zero at a known price W_0 per unit, so that the cost of the input will be $W_0 C(Q)$.

Determine the optimal choice of Q based on the Capital Asset Pricing Model. Can you be sure that the company will choose to produce at all? Are the second-order condition for a maximum satisfied?

How does the optimal Q depend on

i. The risk aversion of the owners of the company?

ii. The riskiness of the output price?

Give a verbal interpretation of your answers.

1(b)

Assume instead that the input factor must be paid for in period one at a price $\tilde{W}$ per unit. This factor price is uncertain as seen from period zero. Maintain the other assumptions from part 1(a). Again determine the optimal choice of Q.

How does the optimal Q depend on

i. The risk aversion of the owners of the company?

ii. The riskiness of the output price?

iii. The riskiness of the factor price?

Give a verbal interpretation of your answers.

1(c)

Assume that in period zero, after the company has announced its plans to start the project, the value of the shares of the company will be equal to the market's valuation of the company's net cash flow from the project in period one.

Under the assumptions in part 1(a): What will be the relationship between the beta of the shares and the similar risk measure for the output price, the beta of $\tilde{P}/V(\tilde{P})$?

Under the assumptions in part 1(b): What will be the relationship between the beta of the shares and the similar risk measures for the output price and the factor price? How does this differ from the preceding answer (the beta from 1(a)) if the factor price has no systematic risk? Interpret the answer. How does it differ if the factor price has the same beta as the output price?

(2)

2(a)

Consider the standard version of the CAPM, illustrated in a (σ, μ) diagram. Assume that the "risk free asset" is really represented by a bank, which offers deposits and loans at the same interest rate r_f. Illustrate in the diagram what kind of preferences will induce a person to borrow from the bank, and what kind of preferences will induce deposits.

2(b)

Assume instead that the bank offers risk free deposits with an interest rate of r_d and risk free loans with an interest rate of r_ℓ, which exceeds r_d. Show that in this situation the investors will divide themselves into three different groups, depositors, borrowers, and some who prefer not to use the bank.

(3)

Consider an economy in which the Capital Asset Pricing Model holds.

(a) Assume that the number of shares is very large, and that an exogenous event makes the value one period ahead of one of the shares more uncertain. More precisely the future value is multiplied by a stochastic variable which has an expected value of unity, and which is stochastically independent of everything else in the economy. Show how this affects the value of the share today, the expected return on the share, and the variance of this return.

(b) Explain why (but not how) the answer to (a) would be different if the share in focus had been a substantial part of the economy.

Exercises for seminar no. 4, 15 October

(1)

Consider a corporation (a limited company, Ltd., Inc., GmbH., ASA) that invests in a production process which only needs input in the form of investment I at time 0, and which produces only at time 1. The produced quantity $Q = f(I)$ is known and fixed after I has been chosen. This Q will be sold at a unit price $\tilde{P}$, which is stochastic as seen from time 0. The company does not borrow and has no other activity. The function f is increasing and concave.

If there are no taxes, the corporation's cash flows at $t = 0, 1$ are $-I$ and $\tilde{P}Q$, and the net value according to the CAPM is

$$-I + \frac{1}{1+r_f}\left(E(\tilde{P}Q) - \lambda \operatorname{cov}(\tilde{P}Q, \tilde{r}_m)\right),$$

where $\lambda = (E(\tilde{r}_m) - r_f)/\sigma_m^2$.

Assume now that the company is taxed at a known rate $\tau \in [0, 1)$. The tax applies to the revenue at $t = 1$, but deductions are given for the investment, with a fraction aI deductible at $t = 0$ and/or a fraction cI deductible at $t = 1$, where a and c are known constants ≥ 0.

For simplicity we assume that the tax values of the deductions, τaI at $t = 0$ and τcI at $t = 1$, are received by the corporation with full certainty. This can, e.g., be the case if the corporation has income from other sources so that it is sure to be in position to pay taxes (and thus "earn" the deductions) in both periods. Or it can follow if the government guarantees payouts of "negative taxes" if deductions exceed the revenues, so that the tax base in some period is negative.

The after-tax cash flows are thus $X_0 = -I(1 - a\tau)$ and $X_1 = \tilde{P}Q(1 - \tau) + cI\tau$, respectively. Define

$$V(\tilde{P}) = \frac{1}{1 + r_f} \left(E(\tilde{P}) - \lambda \operatorname{cov}(\tilde{P}, \tilde{r}_m) \right).$$

(a) Show that the first-order condition for maximization of the net after-tax value can be written

$$f'(I)(1 - \tau)V(\tilde{P}) = 1 - a\tau - \frac{c\tau}{1 + r_f}.$$

Interpret the result: How does the optimal I from this f.o.c. depend on $V(\tilde{P})$ and on a?

(b) Show that, after investment, the beta of the shares in a corporation with only this project depends on the beta of $\frac{\tilde{P}}{V(\tilde{P})}$, which we can call β_P, in the following way:

$$\beta_{X_1} = \frac{V(\tilde{P})f(I)(1 - \tau)}{V(\tilde{P})f(I)(1 - \tau) + c\tau I/(1 + r_f)} \beta_P$$

(We still assume that the tax values of the deductions are earned with full certainty in periods 0 and 1, even though we now consider a corporation with only one project.)

Explain how this formula can be interpreted as a value-weighted average of the betas of separate parts of the cash flow. (What are these betas, values, and weights?)

(c) Consider a cash flow tax, i.e., $a = 1, c = 0$. How does the tax rate influence the optimal I from part (a) above? How does the tax rate influence the beta from part (b) above? (By "how does the tax rate influence," we mean in which direction, if at all.)

(d) Consider next a tax with $a = 0, c = 1 + r_f$. How does the tax rate influence the optimal I from part (a) above? How does the tax rate influence the beta from part (b) above?

(e) Consider finally a tax on gross revenue, $a = 0, c = 0$. How does the tax rate influence the optimal I from part (a) above? How does the tax rate influence the beta from part (b) above?

(2)

Consider the following two alternative stochastic processes, suggested as descriptions of a stock price.

$$P_t = P_{t-1} \cdot u_t \quad \text{with } u_t \text{ i.i.d.} \tag{1}$$

$$P_t = P_0 \cdot e^{a \cdot t} \cdot v_t \quad \text{with } v_t \text{ i.i.d.} \tag{2}$$

You can assume that a is a positive constant, and that $E_{t-1}(v_t) = 1$. You may also, as a simplification, assume that the stock has a beta value of zero.

Discuss whether the two processes have the same expected time path (for some values of a and $E_0(u_t)$). Discuss for each process whether it could describe stock prices in an efficient stock market. If not, how could you device a trading rule to make profits based on observing outcomes? (Moderately difficult: Suggest a rule, and explain in words why it will work. More difficult: Suggest a rule, and prove with calculations that the expected logarithm of the return will represent an excess return in relation to the risk.)

Exercises for seminar no. 5, 29 October

(1)

1(a) Explain the concept "risk free arbitrage opportunity." Explain briefly how this concept is used to prove relationships between the market values of various securities. Show that under some assumptions, the interest rates on two risk free bonds must be the same.

1(b) Assume that the binomial share and option model is valid: Consider a share which for sure does not pay any dividend in the periods we focus on. All agents know that if the share price at time t is S, then the share prices at $t+1$ will be uS or dS. Consider a European call option with two periods left to maturity, when the share price today is $S = 10.00$, the exercise price of the option is $K = 13.19$, the one-period interest rate factor is $e^r = 1.1$, the factor u is 1.2, and the factor d is 1.0. Assume that a price of 0.50 is observed for this option. Show that this creates a risk free arbitrage opportunity, and show how to take advantage of this during the time to the maturity date of the option.

(2)

(Question 2 of exam, spring 1989)

Consider a portfolio consisting of

(i) Two equal European call options on a share which for sure does not pay dividends,

(ii) one short sale of the same share, and

(iii) a number of riskless bonds (unspecified, at this point).

(2a)

Draw a diagram showing the value of such a portfolio at the expiration date of the options. *Based on the diagram*, discuss how the value of the portfolio at a date prior to expiration depends on the uncertainty about the share value at expiration. Make no specific assumptions on the probability distribution of that share value, and restrict yourself to an intuitive argument based on the diagram.

(2b)

Show that at a date prior to expiration, it is possible to obtain an arbitrage profit if the share value exceeds $K + 2C$, where K is the option's exercise price, and C is the option's observed market value at the same date.

(2c)

An individual gets inside information that the corporation which has issued the share, has invested all its funds in a secret, very risky project. The individual knows about the project, but does not know whether it will succeed. Only when the project outcome is known, the project and its outcome will be made public. This will happen before the expiration date of the options. Discuss whether it is possible to make money on this inside information, and whether this possibility is affected by the existence of option markets.

(3)

3(a) Consider a capital market with risk averse agents, in which the standard assumptions of the binomial option value model hold. Assume $e^r = 1.25$, and consider a share with $S_0 = 1.0$, while $\Pr(S_1 = 1.3) = 0.9$ and $\Pr(S_1 = 1.0) = 0.1$. It is certain that the share pays no dividends. A European call option on this share, with $K = 1.15$ and expiration at $t = 1$, sells at $t = 0$ at the price 0.05. How can this arbitrage opportunity be used? What should be sold and bought? What is the arbitrage profit at $t = 0$? What piece of information above is unnecessary?

3(b) Another option is like the first one, but with a different exercise price. Its equilibrium value at $t = 0$ is 0.28. Find the exercise price! (This is a bit tricky.)

(4)

Question 5.27 in Hull, eighth ed., p. 127 (5.24. in seventh ed., p. 125).

Exercises for seminar no. 6, 12 November

(1)

(Will be made available later, in the end of October. Please see course web page.)

(2)

Question 12.17 in Hull, eighth ed., p. 275 (11.17. in seventh ed., p. 258).

(3)

Question 13.16 in Hull, eighth ed., p. 296 (12.15. in seventh ed., p. 274).

(4)

Question 13.17 in Hull, eighth ed., p. 296 (12.16. in seventh ed., p. 274).