The exam consists of three problems. They count equally. Start by reading through the whole exam, and make sure that you allocate time to answering questions you find easy. You can get a good grade even if there are parts of problems that you do not have time to solve.

Problem 1

Consider a person who plans for only one future period, $t = 1$. After consumption today, $t = 0$, has been taken care of, the person has wealth $W > 0$ to invest to provide for consumption, $\tilde{C}$, at $t = 1$. The budget for $\tilde{C}$ consists only of the results from the investment of this wealth. The person has logarithmic utility, i.e., the person maximizes $E[u(\tilde{C})] \equiv E[\ln(\tilde{C})]$. There are only two investment opportunities. A bank offers risk free borrowing and saving at an interest rate r_f. There is also a risky asset with a rate of return $\tilde{r}$.

(a)

Discuss whether we should assume that $\Pr(\tilde{C} > 0) = 1$, i.e., that the consumption at $t = 1$ will be a positive number for sure.

(b)

Assume that $\tilde{r}$ has only two possible outcomes, r_1 and r_2, with $\Pr(\tilde{r} = r_1) = p$. Show that under some assumptions the optimal amount to invest in the risky asset is

$$a^* = \frac{W(1 + r_f)[E(\tilde{r}) - r_f]}{(r_1 - r_f)(r_f - r_2)}.$$

(c)

Give a verbal interpretation of the solution for a^*. You should at least address the following questions: Could the equation give a negative value? What would be the economic interpretation of this? How does uncertainty about $\tilde{r}$ influence the optimal risky investment?

Problem 2

You are asked to sketch two different opportunity sets in the same diagram, and then to consider what can be said about which of those opportunity sets an investor would prefer. All you know about investors is that they are risk averse and their preferences only depend on mean and variance.

Consider a pair of risky securities, 1 and 2, with rates of return $\tilde{r}_j, j = 1, 2$, with the properties $\sigma_1^2 = \text{var}(\tilde{r}_1) = 0.2^2 = 0.04$, $\sigma_2^2 = \text{var}(\tilde{r}_2) = 0.4^2 = 0.16$, and $\sigma_{12} = \text{cov}(\tilde{r}_1, \tilde{r}_2) = 0.01$. Two additional pieces of information are needed in order to define the opportunity set of portfolios from the two securities: Short selling may or may not be allowed (see below). Furthermore, with regard to the expected rates of return, $\mu_1 = E(\tilde{r}_1)$ and $\mu_2 = E(\tilde{r}_2)$, you are asked to consider two cases:

Case (i): $\mu_1 = 0.06$, $\mu_2 = 0.18$.

Case (ii): $\mu_1 = 0.14$, $\mu_2 = 0.02$.

(a)

For case (i), find that portfolio of the two securities which has a rate of return with the lowest variance. Show that this variance is $0.035 \approx 0.187^2$. Explain why the same result holds for case (ii). Sketch the opportunity sets for the two cases in the same diagram.

(b)

Can you determine whether an investor would prefer case (i) to case (ii) when short selling is not allowed?

(c)

Can you determine whether an investor would prefer case (i) to case (ii) when short selling is allowed?

Problem 3

Consider a call option (American or European) with expiration at date T on a share which for sure does not pay dividends in the time interval $(0, T)$.

The price at time 0 of the share is S, and the option's exercise price is K. The interest rate on riskless bonds is r.

(a)

Explain why we can assume that the option value is greater than or equal to $S - Ke^{-rt}$.

(b)

Explain how the result in (a) can be used to show that the value of an American call option of the type described has the same value as a call option that is European, but otherwise identical.

(c)

Explain why the explanations above do not hold if there is a possibility that the stock pays dividends at some specific dates between time 0 and T. Explain why it is not optimal to exercise the American option for most of the time between 0 and T.

SUGGESTED ANSWERS

Problem 1

Consider a person who plans for only one future period, $t = 1$. After consumption today, $t = 0$, has been taken care of, the person has wealth $W > 0$ to invest to provide for consumption, $\tilde{C}$, at $t = 1$. The budget for $\tilde{C}$ consists only of the results from the investment of this wealth. The person has logarithmic utility, i.e., the person maximizes $E[u(\tilde{C})] \equiv E[\ln(\tilde{C})]$. There are only two investment opportunities. A bank offers risk free borrowing and saving at an interest rate r_f. There is also a risky asset with a rate of return $\tilde{r}$.

(a)

Discuss whether we should assume that $\Pr(\tilde{C} > 0) = 1$, i.e., that the consumption at $t = 1$ will be a positive number for sure.

ANSWER

There are two reasons why a negative C does not make sense here: It is mathematically impossible to take the logarithm of a negative number, and it is also difficult to see what it should mean to let someone have negative consumption. Thus we may assume that a negative C is not allowed in the problem, or at least that the solution will have zero probability of such an outcome. The particular utility function, $\ln(C)$, has the property that it goes to minus infinity when C approaches zero from above, and that marginal utility goes to infinity. Thus the person will want to avoid the possibility of a negative C at all costs. If a solution with zero probability of $C < 0$ can be obtained, we need not impose an extra assumption ruling out $C < 0$, since this will be chosen by the person. With a strictly positive W and a risk free interest rate greater than minus one hundred percent, such a solution can be obtained.

(b)

Assume that $\tilde{r}$ has only two possible outcomes, r_1 and r_2, with $\Pr(\tilde{r} = r_1) = p$. Show that under some assumptions the optimal amount to invest in the risky asset is

$$a^* = \frac{W(1+r_f)[E(\tilde{r}) - r_f]}{(r_1 - r_f)(r_f - r_2)}.$$

ANSWER

The budget for $t = 1$ consumption is

$$\tilde{C} = W(1+r_f) + a(\tilde{r} - r_f).$$

The first-order condition for maximizing the expected utility of this w.r.t. a is

$$E\left[\frac{\tilde{r} - r_f}{W(1+r_f) + a(\tilde{r} - r_f)}\right] = 0.$$

With only two possible outcomes, this reduces to

$$\frac{p(r_1 - r_f)}{W(1+r_f) + a(r_1 - r_f)} = \frac{(1-p)(r_f - r_2)}{W(1+r_f) + a(r_2 - r_f)}.$$

This can be rearranged to the equation in the text.

If the denominator $(r_1 - r_f)(r_f - r_2)$ is not positive, the risky asset either dominates or is dominated by the risk free asset (in the first-order stochastic sense). We must assume that such dominance does not occur. If it had occurred, a corner solution for a^* would be optimal. If infinite borrowing and/or short sales are allowed, the "corner" could be infinity.

We should also check that the worst outcome for C is positive. (If not, the logarithmic utility function is not well defined.) If r_2 is the smaller of the two outcomes for $\tilde{r}$, we need

$$W(1+r_f) + a(r_2 - r_f) > 0.$$

If the risky asset is not dominate the risk free, we need $r_2 < r_f$, so the inequality above gives a limitation on the size of a, namely

$$a < \frac{W(1+r_f)}{r_f - r_2}.$$

But the expression we found for a^* always satisfies this, since we know that $r_1 - r_f > E(\tilde{r}) - r_f$. Thus we need no additional assumption to secure $C > 0$.

(c)

Give a verbal interpretation of the solution for a^*. You should at least address the following questions: Could the equation give a negative value? What would be the economic interpretation of this? How does uncertainty about $\tilde{r}$ influence the optimal risky investment?

ANSWER

If we assume there is no dominance, so that the denominator is positive, there could be a negative value if $E(\tilde{r}) < r_f$. This may happen. Short sale is desired. If not allowed, $a = 0$ will be optimal. The interpretation is the feature known from other models as well, known as local risk neutrality: $E(\tilde{r}) > r_f$ is necessary and sufficient to induce some risk taking. Uncertainty about $\tilde{r}$ is in this particular model reflected in the dispersion of the outcomes, only, appearing in the denominator in the a^* expression. More uncertainty means a higher denominator and a lower a^*.

Problem 2

You are asked to sketch two different opportunity sets in the same diagram, and then to consider what can be said about which of those opportunity sets an investor would prefer. All you know about investors is that they are risk averse and their preferences only depend on mean and variance.

Consider a pair of risky securities, 1 and 2, with rates of return $\tilde{r}_j, j = 1, 2$, with the properties $\sigma_1^2 = \text{var}(\tilde{r}_1) = 0.2^2 = 0.04$, $\sigma_2^2 = \text{var}(\tilde{r}_2) = 0.4^2 = 0.16$, and $\sigma_{12} = \text{cov}(\tilde{r}_1, \tilde{r}_2) = 0.01$. Two additional pieces of information are needed in order to define the opportunity set of portfolios from the two securities: Short selling may or may not be allowed (see below). Furthermore, with regard to the expected rates of return, $\mu_1 = E(\tilde{r}_1)$ and $\mu_2 = E(\tilde{r}_2)$, you are asked to consider two cases:

Case (i): $\mu_1 = 0.06$, $\mu_2 = 0.18$.

Case (ii): $\mu_1 = 0.14$, $\mu_2 = 0.02$.

(a)

For case (i), find that portfolio of the two securities which has a rate of return with the lowest variance. Show that this variance is $0.035 \approx 0.187^2$. Explain why the same result holds for case (ii). Sketch the opportunity sets for the two cases in the same diagram.

ANSWER

The formula for the variance-minimizing portfolio weight on security 1 is derived in p. 6 of the lecture notes of 31 Jan 2011:

$$a_{\text{min}} = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}.$$

In our problem, the numerical value is $\frac{0.16-0.01}{0.04+0.16-0.02} = \frac{5}{6}$. The resulting variance is

$$\left(\frac{5}{6}\right)^2 \cdot 0.04 + \left(\frac{1}{6}\right)^2 \cdot 0.16 + 2 \cdot \frac{5}{6} \cdot \frac{1}{6} \cdot 0.01 = \frac{1.26}{36} = 0.035.$$

Clearly, there are not expected rates of return involved in these calculations, so they valid for both cases. The expected rates of return of the two variance-minimizing portfolios are, in both cases, given as

$$\frac{5}{6}\mu_1 + \frac{1}{6}\mu_2,$$

which takes on the two values 0.08 (case 1) and 0.12 (case 2). By plotting the locations of the two securities themselves (which correspond to $a = 1$ and $a = 0$, respectively), and these two minimum (or leftmost) points, one gets three points for each case, which allow a sketch of each hyperbola. See separate diagram, showing those parts of the hyperbolae which are available without short sales. Case (i) is Series 1, case (ii) is Series 2.

(b)

Can you determine whether an investor would prefer case (i) to case (ii) when short selling is not allowed?

ANSWER

When short selling is not allowed, one cannot determine which opportunity set is more favorable to an investor without knowing more about his/her preferences. For high risk aversion, case (ii) is attractive. But for lower risk aversion, there is the possibility that the preferred point is for $\mu > 0.14$, which can only be obtained from case (i), although at the cost of higher risk.

(c)

Can you determine whether an investor would prefer case (i) to case (ii) when short selling is allowed?

ANSWER

In this case, the opportunity sets extend towards plus and minus infinity along the hyperbolae. Case (ii) is always better.

Problem 3

Consider a call option (American or European) with expiration at date T on a share which for sure does not pay dividends in the time interval $(0, T)$. The price at time 0 of the share is S, and the option's exercise price is K. The interest rate on riskless bonds is r.

(a)

Explain why we can assume that the option value is greater than or equal to $S - Ke^{-rt}$.

ANSWER

See lecture notes for 21 March 2011, p. 9.

(b)

Explain how the result in (a) can be used to show that the value of an American call option of the type described has the same value as a call option that is European, but otherwise identical.

ANSWER

See lecture notes for 21 March 2011, p. 9–10.

(c)

Explain why the explanations above do not hold if there is a possibility that the stock pays dividends at some specific dates between time 0 and T. Explain why it is not optimal to exercise the American option for most of the time between 0 and T.

ANSWER

See lecture notes for 21 March 2011, p. 11.

More inequality results on option values

Absence-of-arbitrage proofs for American calls:

1. $C \geq 0$: If not, buy option, keep until expiration. Get something positive now, certainly nothing negative later.
2. $C \leq S$: If not, buy share, sell (i.e., write) call, receive $C - S > 0$. Get $K > 0$ if option is exercised, get S if not.
3. $C \geq S - K$: If not, buy option, exercise immediately.
4. When (for sure) no dividends: $C \geq S - Ke^{-rt}$: If not, do the following:

			Expiration	
	Now	Div. date	If $S_T \leq K$	If $S_T > K$
Sell share	S	0	$-S_T$	$-S_T$
Buy call	$-C$	0	0	$S_T - K$
Lend	$-Ke^{-rt}$	0	K	K
	≥ 0	0	≥ 0	0

A riskless arbitrage.

Important implication: *American call option on shares which certainly will not pay dividends before option's expiration, should not be exercised before expiration,* since

$$C \geq S - Ke^{-rt} > S - K.$$

Worth more "alive than dead." When no dividends: *Value of American call equal to value of European.*

Summing up some results

Both American and European call options on shares which for sure pay no dividends:

$$C \geq S - Ke^{-rt} > S - K.$$

American call options on shares which may pay dividends:

$$C \geq S - K.$$

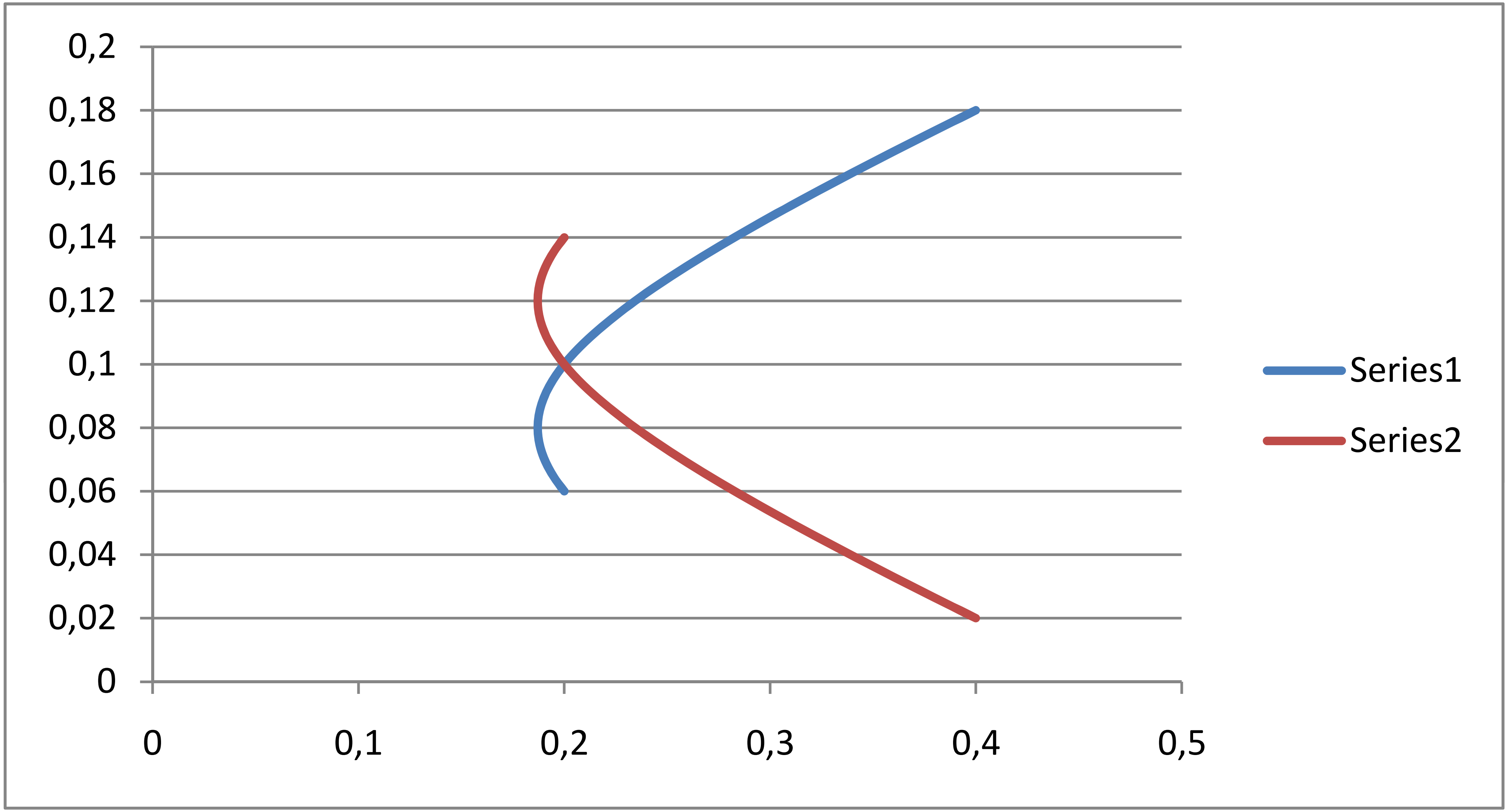
American calls when dividends possible: More

- For each dividend payment: Two dates.
  - One date for announcement, after which D known.
  - One *ex-dividend* date, after which share does not give the right to that dividend payment.
- Our interest is in ex-dividend dates.
- Owners of shares on morning of ex-div. date receive D.
- Assume there is a given number of ex-div. dates.
- Say, 2 ex-div. dates, t_{d1}, t_{d2}, before option's expiration, T.
- Can show: $C > S - K$ except just before t_{d1}, t_{d2}, T.
- Assume contrary, $C \leq S - K$. Then riskless arbitrage:
- Buy call, exercise just before:

	Now	Just before next t_{di} or T
Buy call	$-C$	$S - K$
Sell share	S	$-S$
Lend	$-K$	$Ke^{r\Delta t}$
	≥ 0	$K(e^{r\Delta t} - 1)$

- Riskless arbitrage, except if $\Delta t \approx 0$, just before.

Implication: *When possible ex-dividend dates are known, American call options should never be exercised except perhaps just before one of these, or at expiration.*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