The exam consists of three problems. They count equally. Start by reading through the whole exam, and make sure that you allocate time to answering questions you find easy. You can get a good grade even if there are parts of problems that you do not have time to solve.

You are warned that there may be information given which is not necessary to answer the problems.

Problem 1

(a)

Explain what is meant by *risk aversion* and by the *measure of absolute risk aversion.*

(b)

Consider the equation

$$E\left[U(Y+\tilde{Z})\right] = U\left(Y + E(\tilde{Z}) - \Pi(Y, \tilde{Z})\right).$$

Define the variables in the equation and explain in particular what is the meaning of $\Pi(Y, \tilde{Z})$. Give a verbal explanation of why and how U and Y may affect $\Pi(Y, \tilde{Z})$.

(c)

Under some assumption(s) there is a particular relation between the measure of absolute risk aversion and $\Pi(Y, \tilde{Z})$. State (i.e., write down) and explain this relation, the assumption(s) behind it, and explain what other variable(s) appear(s) in it.

(d)

Show mathematically how to derive the relation mentioned in (c).

Problem 2

Consider an economy where the assumptions behind the standard Capital Asset Pricing Model hold for the period between time zero and time one. There are many assets and many agents, but no taxes. The risk free interest rate is r_f, and the rate of return of the market portfolio is $\tilde{r}_m$, with expectation $\bar{r}_m = E(\tilde{r}_m) > r_f$ and variance $\sigma_m^2 = \text{var}(\tilde{r}_m) > 0$. Consider a corporation that will have a cash flow $\tilde{X}$ at time 1. The corporation has no debt. The cash flow has expectation $\bar{X} = E(\tilde{X})$, variance $\sigma_X^2 = \text{var}(\tilde{X}) > 0$, and covariance $\sigma_{Xm} = \text{cov}(\tilde{X}, \tilde{r}_m) > 0$ with the rate of return on the market portfolio.

(a)

Write down a formula for the value of the shares in the corporation at time zero as a function of some or all of the parameters mentioned in the introduction. Give a verbal interpretation of how the value is affected by the riskiness of $\tilde{X}$.

(b)

Define the beta value of the shares in the corporation. How is this beta value affected by an increase in σ_X^2 (assuming that the increase is perceived at time zero)? How is the beta value affected by an increase in σ_{Xm} (likewise, perceived at time zero)? You may analyze each of these changes as if the other variable is unchanged, i.e., first, σ_X^2 is increased while σ_{Xm} is unchanged, then vice versa, that σ_{Xm} is increased while σ_X^2 is unchanged.

(c)

The CAPM is often illustrated in a $(\sigma, E(\tilde{r}))$ diagram with a curve known as the frontier portfolio set and a line (or a ray, to be precise) known as the capital market line. Assume that the beta of the shares is around 1.2. Sketch the typical location of the corporation in the diagram before the increases mentioned in part (b). Then show the effects on this location of each of these two increases, separately.

Problem 3

Assume that the binomial share and option model is valid. Consider a share which for sure does not pay any dividend in the periods we focus on. All agents know that if the share price at time t is S_t, then the share price at $t+1$ will be uS_t or dS_t. Consider a European put option on this share with two periods left to maturity, when the share price today (time zero) is $S_0 = 10.00$, the exercise price of the option is $K = 8.82$, the one-period interest rate factor is $e^r = 1.1$, the factor u is 1.3, and the factor d is 0.8. The probability of $(S_{t+1}/S_t = u)$ is $p^* = 0.8$.

Since calculators are not allowed for this exam, there is a table at the end which provides some calculations that may be useful for answering the questions that follow. The table also contains some additional calculations which are not needed, in order that the correct methods for answering the questions shall not be too obvious.

(a)

Find the values of the option at the maturity date for the different possible outcomes of S_2. Show that the market value of the option at time zero, using the risk-neutral valuation model, is 0.32.

(b)

Verify that the market value of the option at time zero, based on a replicating portfolio strategy, is also 0.32.

(c)

Assume that the option is instead traded at the price $p_0 = 0.42$ at time zero. Show that this creates an arbitrage opportunity. Show how to take advantage of the opportunity: What should be bought and sold at time zero? For each of the possible observations of the share price at time one, what should be bought and sold at that time?

Table of calculations; some of these may be useful

$0.605 \cdot 13$	=	7.865	$2.42/4$	=	0.605
$7.865/1.1$	=	7.15	$0.6 \cdot 0.6$	=	0.36
$6.292/1.1$	=	5.72	$0.6 \cdot 0.4$	=	0.24
$0.605 \cdot 8$	=	4.84	$0.42 \cdot 0.42$	=	0.1764
$13 \cdot 0.16$	=	2.08	$0.88/5$	=	0.176
$2.42/1.21$	=	2	$0.4 \cdot 0.4$	=	0.16
$1.3 \cdot 1.3$	=	1.69	$0.8 \cdot 0.2$	=	0.16
$1.1 \cdot 1.1$	=	1.21	$0.176/1.1$	=	0.16
$1.3 \cdot 0.8$	=	1.04	$0.32 \cdot 0.32$	=	0.1024
$0.1764/0.24$	=	0.735	$0.32 \cdot 0.176$	=	0.05632
$0.8 \cdot 0.8$	=	0.64	$0.2 \cdot 0.2$	=	0.04

Answers

Problem 1

(a)

Explain what is meant by *risk aversion* and by the *measure of absolute risk aversion.*

> Definitions are found in Danthine and Donaldson (2nd ed.), section 4.2. The concepts are only defined in relation to expected utility, and in this course, the U function is assumed to be twice differentiable. In the lecture notes the definitions are given on pp. 17 and 18 for 19 January,
>
> http://www.uio.no/studier/emner/sv/oekonomi/ECON4510/v12/undervisningsmateriale/lect1901.pdf

(b)

Consider the equation

$$E\left[U(Y+\tilde{Z})\right] = U\left(Y + E(\tilde{Z}) - \Pi(Y,\tilde{Z})\right).$$

Define the variables in the equation and explain in particular what is the meaning of $\Pi(Y,\tilde{Z})$. Give a verbal explanation of why and how U and Y may affect $\Pi(Y,\tilde{Z})$.

> The equation is (4.6b) in Danthine and Donaldson.
>
> The left-hand side is an expression for expected utility, with the argument, $Y+\tilde{Z}$ being a stochastic income (or wealth, consumption budget). The right-hand side is an expression for utility, with the argument, $Y + E(\tilde{Z}) - \Pi(Y,\tilde{Z})$ being an income to be received with full certainty. Y is some non-stochastic income, and $\tilde{Z}$ is a stochastic addition to Y. In D&D, there is the subsequent assumption that $E(\tilde{Z}) = 0$.
>
> On the right-hand side $\tilde{Z}$ is replaced by its expected value $E(\tilde{Z})$, and this will only result in the same utility (as shown by the given equation) for a risk averse individual if some amount is subtracted on the right-hand side. This is the amount $\Pi(Y,\tilde{Z})$, which is

known as the risk premium. It tells how much the individual is willing to pay (i.e., to give up, in terms of consumption budget at the same time) to avoid the uncertainty in $\tilde{Z}$, i.e., in order to have $\tilde{Z}$ replaced by its expected value.

In general, we must assume that the risk premium Π will differ between different people with different U functions. More risk aversion will typically mean higher values of Π. Also, for a given U function, we must assume that Π will depend on the income level, expressed by Y. This can be explained by the fact that (absolute) risk aversion typically depends on the income level, except for the special class of CARA utility functions.

(c)

Under some assumption(s) there is a particular relation between the measure of absolute risk aversion and $\Pi(Y, \tilde{Z})$. State (i.e., write down) and explain this relation, the assumption(s) behind it, and explain what other variable(s) appear(s) in it.

The assumption needed is that the variability in $\tilde{Z}$ is not too large, or equivalently, that the risk premium is small. This allows us to ignore higher-order term in the Taylor expansion of the equation defining Π. We find that the risk premium is approximately proportional to the measure of absolute risk aversion. More precisely,

$$\Pi(Y, \tilde{Z}) \approx \frac{1}{2}\sigma_Z^2 R_A(Y),$$

where $\sigma_Z^2 \equiv \text{var}(\tilde{Z})$, and $R_A(Y)$ is the measure of absolute risk aversion. It is natural that the risk premium depends on the variability of $\tilde{Z}$, and the result shows that the effects of this and of the individual's risk aversion separate nicely and appear multiplicatively.

(d)

Show mathematically how to derive the relation mentioned in (c).

The derivation is shown in the above-mentioned lecture notes, p. 20, and in D&D, p. 63f.

Problem 2

Consider an economy where the assumptions behind the standard Capital Asset Pricing Model hold for the period between time zero and time one. There are many assets and many agents, but no taxes. The risk free interest rate is r_f, and the rate of return of the market portfolio is $\tilde{r}_m$, with expectation $\bar{r}_m = E(\tilde{r}_m) > r_f$ and variance $\sigma_m^2 = \text{var}(\tilde{r}_m) > 0$. Consider a corporation that will have a cash flow $\tilde{X}$ at time 1. The corporation has no debt. The cash flow has expectation $\bar{X} = E(\tilde{X})$, variance $\sigma_X^2 = \text{var}(\tilde{X}) > 0$, and covariance $\sigma_{Xm} = \text{cov}(\tilde{X}, \tilde{r}_m) > 0$ with the rate of return on the market portfolio.

(a)

Write down a formula for the value of the shares in the corporation at time zero as a function of some or all of the parameters mentioned in the introduction. Give a verbal interpretation of how the value is affected by the riskiness of $\tilde{X}$.

> There is no explicit question about how the formula can be derived. The derivation is shown in lecture notes 9 February 2012 p. 6, and in p. 126 of D&D, but this derivation (the intermediate steps) is not an essential part of the answer. It would be a plus, however, if the candidates explains verbally how the rate of return, that should satisfy the CAPM, is related to the cash flow. The explanation could go like this:
>
> Since there is no debt and no taxes, it is assumed that the total value of shares in the corporation at time 1 is equal to $\tilde{X}$, including dividends paid out, if any. If $V(\tilde{X})$ is the total value at time zero of these shares, then the rate of return, $\tilde{X}/V(\tilde{X}) - 1$ should satisfy the CAPM, which implies that
>
> $$V(\tilde{X}) = \frac{1}{1 + r_f} \left(\bar{X} - \lambda \sigma_{Xm} \right),$$
>
> where
>
> $$\lambda \equiv \frac{\bar{r}_m - r_f}{\sigma_m^2}.$$

Only one aspect of the riskiness of $\tilde{X}$ appears in the equation, the covariance with the rate of return on the market portfolio. This is just another version of the main insight in the CAPM, that only non-diversifiable risk matters for the pricing of assets.

(b)

Define the beta value of the shares in the corporation. How is this beta value affected by an increase in σ_X^2 (assuming that the increase is perceived at time zero)? How is the beta value affected by an increase in σ_{Xm} (likewise, perceived at time zero)? You may analyze each of these changes as if the other variable is unchanged, i.e., first, σ_X^2 is increased while σ_{Xm} is unchanged, then vice versa, that σ_{Xm} is increased while σ_X^2 is unchanged.

The definition of the beta of shares is

$$\beta_{rx} = \frac{\operatorname{cov}\left(\frac{\tilde{X}}{V(\tilde{X})}, \tilde{r}_m\right)}{\sigma_m^2},$$

which can be rewritten as

$$\beta_{rx} = \frac{\operatorname{cov}\left(\tilde{X}, \tilde{r}_m\right)}{V(\tilde{X})\sigma_m^2}.$$

The first question can be answered directly from the observation from part (a) that $V(\tilde{X})$ does not change. With unchanged covariance (the numerator), the beta is also unchanged.

The second question also has an unambiguous answer, since part (a) shows that $V(\tilde{X})$ decreases when the covariance in the numerator increases. The overall effect is thus to increase β_{rx}. A common mistake may be to disregard the change in $V(\tilde{X})$, which gives the right conclusion, but is nevertheless a somewhat deficient answer.

That σ_X^2 and σ_{Xm} can be treated like this, changing one and holding the other constant, is known from the lecture notes, 9 February 2012, p. 8.

(c)

The CAPM is often illustrated in a $(\sigma, E(\tilde{r}))$ diagram with a curve known as the frontier portfolio set and a line (or a ray, to be precise) known as the capital market line. Assume that the beta of the shares is around 1.2. Sketch the typical location of the corporation in the diagram before the increases mentioned in part (b). Then show the effects on this location of each of these two increases, separately.

> The location could be on the frontier portfolio set, but is typically to the right of it. I.e., there is no reason to believe that shares in a particular corporation are mean-variance efficient — this would not be typical. With a beta somewhat above unity, the vertical position is higher up in the diagram than the tangency point at the market portfolio.
>
> An increase in σ_X^2 has no effect on the beta of the shares, and thus no effect on the expected rate of return. There is no move in the vertical direction. Since the rate of return on the shares is $\tilde{X}/V(\tilde{X}) - 1$, and $V(\tilde{X})$ is unchanged (cf. part (b)) while the variance of $\tilde{X}$ is increased, there will be an increase in the variance of the rate of return. Thus there is a movement to the right in the diagram.
>
> The effect of an increase in σ_{Xm} is to increase the beta of the shares, cf. part (b) above. This means that the expected rate of return after this increased covariance will be higher than before. This will be the situation after (and not including) an immediate drop in the value of the shares. So there is a movement upwards in the diagram. Since the rate of return on the shares is $\tilde{X}/V(\tilde{X})-1$, and $V(\tilde{X})$ is decreased while the variance of $\tilde{X}$ is unchanged by assumption, there will be an increase in the variance of the rate of return. Thus there is a movement to the right in the diagram.

Problem 3

Assume that the binomial share and option model is valid. Consider a share which for sure does not pay any dividend in the periods we focus on. All agents know that if the share price at time t is S_t, then the share price at

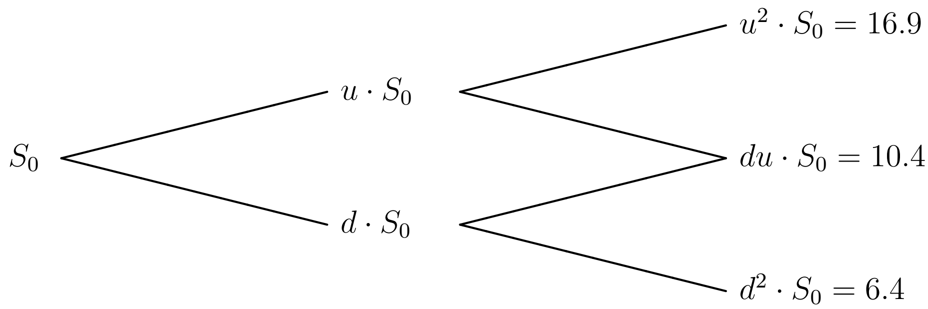
$t + 1$ will be uS_t or dS_t. Consider a European put option on this share with two periods left to maturity, when the share price today (time zero) is $S_0 = 10.00$, the exercise price of the option is $K = 8.82$, the one-period interest rate factor is $e^r = 1.1$, the factor u is 1.3, and the factor d is 0.8. The probability of $(S_{t+1}/S_t = u)$ is $p^* = 0.8$.

Since calculators are not allowed for this exam, there is a table at the end which provides some calculations that may be useful for answering the questions that follow. The table also contains some additional calculations which are not needed, in order that the correct methods for answering the questions shall not be too obvious.

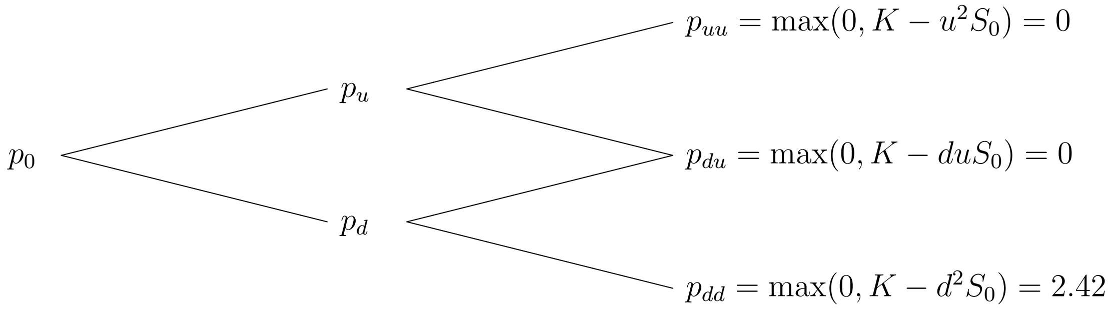
(a)

Find the values of the option at the maturity date for the different possible outcomes of S_2. Show that the market value of the option at time zero, using the risk-neutral valuation model, is 0.32.

The tree which describes the possible outcomes of the share value looks like:



The corresponding tree for the option looks like:



The probability of moving up is $p^* = 0.8$ in the real world, but this information is not needed. (Cf. the introduction to the exam.) In the "risk neutral world" the probability is given by

$$p = \frac{e^r - d}{u - d} = \frac{0.3}{0.5} = 0.6.$$

The risk-adjusted (also known as risk-neutral) probability for the dd outcome is thus $0.4^2 = 0.16$, and the present value of the risk-adjusted expected value of the option is $e^{-2r} \cdot 0.16 \cdot 2.42 = 0.32$. This is thus the market value using the method of risk-neutral valuation.

(b)

Verify that the market value of the option at time zero, based on a replicating portfolio strategy, is also 0.32.

We need a replicating portfolio for the option at each node of the tree at $t = 0, 1$. These three nodes are $(0, u, d)$. Such a portfolio consists of a number of shares, Δ, and an amount invested in bonds, B. At node u the option will for sure expire worthless one period later, so the portfolio is zero. At node d, we know that (Δ_d, B_d) must satisfy

$$d^2 S_0 \Delta_d + B_d e^r = 2.42$$

and

$$duS_0 \Delta_d + B_d e^r = 0,$$

with solution

$$\Delta_d = -0.605 \quad \text{and} \quad B_d = 5.72,$$

with total value equal to $\Delta_d S_0 d + B_d = 0.88$. At node 0, we know that (Δ_0, B_0) must satisfy

$$dS_0\Delta_0 + B_0 e^r = 0.88$$

and

$$uS_0\Delta_0 + B_0 e^r = 0,$$

with solution

$$\Delta_0 = -0.176 \quad \text{and} \quad B_0 = 2.08.$$

The total value is $\Delta_0 S_0 + B_d = 0.32$, as was to be shown.

(c)

Assume that the option is instead traded at the price $p_0 = 0.42$ at time zero. Show that this creates an arbitrage opportunity. Show how to take advantage of the opportunity: What should be bought and sold at time zero? For each of the possible observations of the share price at time one, what should be bought and sold at that time?

The arbitrage opportunity which follows from a too high observed price is to buy the replicating portfolio and sell, short sell, or issue the option. This brings in $0.42 - 0.32$ in arbitrage profits at time zero for each unit of the transaction. As long as prices do not change, the transaction will be profitable in any amount. An investor would want to undertake infinitely many of these transactions as long as prices are unchanged.

If the upper node is the outcome at time one, the replicating portfolio (Δ_0, B_0) has a net value of zero, and should be closed down.

If the lower node is the outcome at time one, the replicating portfolio (Δ_0, B_0) has a net value of 0.88. This is exactly sufficient to buy the new replicating portfolio (Δ_d, B_d) at that node.

If the middle node is the outcome at time two, the replicating portfolio (Δ_d, B_d) has a net value of zero, and should be closed down.

If the lower node dd is the outcome at time two, the replicating portfolio (Δ_d, B_d) has a net value of 2.42. This is exactly sufficient to pay off the owner of the put option, or the net addition needed if the share is to be bought at price 8.82 (according to the option) and sold to the market at price 6.4.

Table of calculations; some of these may be useful

$0.605 \cdot 13$	=	7.865	$2.42/4$	=	0.605
$7.865/1.1$	=	7.15	$0.6 \cdot 0.6$	=	0.36
$6.292/1.1$	=	5.72	$0.6 \cdot 0.4$	=	0.24
$0.605 \cdot 8$	=	4.84	$0.42 \cdot 0.42$	=	0.1764
$13 \cdot 0.16$	=	2.08	$0.88/5$	=	0.176
$2.42/1.21$	=	2	$0.4 \cdot 0.4$	=	0.16
$1.3 \cdot 1.3$	=	1.69	$0.8 \cdot 0.2$	=	0.16
$1.1 \cdot 1.1$	=	1.21	$0.176/1.1$	=	0.16
$1.3 \cdot 0.8$	=	1.04	$0.32 \cdot 0.32$	=	0.1024
$0.1764/0.24$	=	0.735	$0.32 \cdot 0.176$	=	0.05632
$0.8 \cdot 0.8$	=	0.64	$0.2 \cdot 0.2$	=	0.04