The exam consists of three problems. They count equally. Start by reading through the whole exam, and make sure that you allocate time to answering questions you find easy. You can get a good grade even if there are parts of problems that you do not have time to solve.

You are warned that there may be information given which is not necessary to answer the problems.

Problem 1

(a)

Consider the function $U(C) = aC^b$, with a and b being constant, real numbers. The function is only defined for $C \geq 0$. What conditions must be satisfied for $E[U(C)]$ to represent the utility of a von Neumann-Morgenstern type person who is risk averse? What are the measures of absolute and relative risk aversion for this utility function?

Answer

We have $U'(C) = abC^{b-1}$, which must be positive, and $U''(C) = ab(b-1))C^{b-2}$, which must be negative. The first requirement means $ab > 0$, and taking this into account, the second means $b < 1$. We can have either $a < 0, b < 0$ or $a > 0, b \in (0, 1)$. The two measures of risk aversion are

$$R_A(C) = \left|\frac{U''(C)}{U'(C)}\right| = \frac{1-b}{C} \quad \text{and} \quad R_R(C) = \left|\frac{U''(C)C}{U'(C)}\right| = 1 - b.$$

(We observe that the relative risk aversion is constant, but there is no question about this.)

(b)

Assume that the person with the utility function mentioned in part (a) is planning for only one future period, $t = 1$. (There is no reason to discuss how much will be consumed at $t = 0$.) The person has wealth $W > 0$ to invest to provide for future consumption, $\tilde{C}$, at $t = 1$. The budget for $\tilde{C}$ consists only of the results from the investment of this wealth. There are only two investment opportunities. A bank offers risk free borrowing and saving at an interest rate r_f. There is also a risky asset with a rate of return $\tilde{r}$. Formulate the person's maximization problem for the future period, and write down the first-order condition for a maximum.

Answer

The budget for consumption is

$$\tilde{C} = (W - v)(1 + r_f) + v(1 + \tilde{r}) = W(1 + r_f) + v(\tilde{r} - r_f).$$

The maximization problem is

$$\max_v E\left\{a\left[W(1 + r_f) + v(\tilde{r} - r_f)\right]^b\right\}.$$

The first-order condition is

$$E\left\{ab\left[W(1 + r_f) + v(\tilde{r} - r_f)\right]^{b-1}(\tilde{r} - r_f)\right\} = 0.$$

(c)

Assume that $\tilde{r}$ has only two possible outcomes, r_1 and r_2, with numbers chosen such that $r_1 > r_2$, and with $\Pr(\tilde{r} = r_1) = p$. Show that under some assumptions, the optimal amount to invest in the risky asset is

$$v^* = \frac{W(1 + r_f)(X - 1)}{r_1 - r_f + X(r_f - r_2)},$$

where X is defined by

$$X = \left[\frac{p(r_1 - r_f)}{(1 - p)(r_f - r_2)}\right]^{\frac{1}{1-b}}.$$

One assumption you will need is $r_2 < r_f < r_1$. Give an economic interpretation of this assumption and its implications.

Answer

With only two outcomes, the maximization problem is reduced to

$$\max_v \left\{pa\left[W(1 + r_f) + v(r_1 - r_f)\right]^b + (1 - p)a\left[W(1 + r_f) + v(r_2 - r_f)\right]^b\right\}.$$

The first-order condition is now

$$pab\left[W(1+r_f)+v(r_1-r_f)\right]^{b-1}(r_1-r_f)$$

$$+(1-p)ab\left[W(1+r_f)+v(r_2-r_f)\right]^{b-1}(r_2-r_f)=0.$$

This can be rearranged to

$$pab\left[W(1+r_f)+v(r_1-r_f)\right]^{b-1}(r_1-r_f)$$

$$=(1-p)ab\left[W(1+r_f)+v(r_2-r_f)\right]^{b-1}(r_f-r_2).$$

We observe that all factors are positive (if they are well defined), except perhaps the last factors on each side of the equality. With $r_1 > r_f > r_2$, they will also be positive, and this is clearly a necessary condition for an interior solution. If not, i.e., if both outcomes of $\tilde{r}$ are either above or below r_f, we would have a case of stochastic dominance of first order. The condition also implies that the fraction in square brackets in the definition of X is positive.

Rearrange now: Divide through by $\left[W(1+r_f)+v(r_2-r_f)\right]^{b-1}(r_1-r_f)apb$, and then raise both sides of the equation to the power $1/(b-1)$. This gives the equation

$$\frac{W(1+r_f)+v(r_1-r_f)}{W(1+r_f)+v(r_2-r_f)}=\left[\frac{(1-p)(r_f-r_2)}{p(r_1-r_f)}\right]^{\frac{1}{b-1}}=X$$

which can be solved for v to find the equation asked for. Observe that X is positive, but not necessarily greater than 1. Observe also that the denominator in the given v^* equation is positive for any value of X (since it is increasing in X and positive even if X takes on the value at its lower limit, which is zero). The sign of v^* is thus equal to the sign of $X-1$.

We should check that the worst outcome for C is positive. (If not, the utility function is not well defined.) Since r_2 is the smaller of the two outcomes for $\tilde{r}$, we need

$$W(1+r_f)+v(r_2-r_f)>0.$$

Since $r_2 < r_f$, the inequality above gives a limitation on the size of v, namely

$$\frac{W(1+r_f)}{r_f-r_2}>v.$$

But the expression we found for v^* always satisfies this. This can be found by plugging it in, and rearranging:

$$\frac{W(1+r_f)}{r_f - r_2} > \frac{W(1+r_f)(X-1)}{r_1 - r_f + X(r_f - r_2)}.$$

Multiply by $[r_1 - r_f + X(r_f - r_2)](r_f - r_2)/(W(1+r_f))$ on both sides to get the equivalent inequality

$$r_1 - r_f + X(r_f - r_2) \equiv r_1 - r_2 + (X-1)(r_f - r_2) > (X-1)(r_f - r_2),$$

which is clearly true.

Thus we need no additional assumption to secure $C > 0$.

(d)

Based on the assumptions in (c), show that if $E(\tilde{r}) = r_f$, then $v^* = 0$. With this situation as a starting point, what would be the effect on v^* of an increase in p? Give an economic interpretation of this effect.

Answer

The condition can be rearranged:

$$E(\tilde{r}) \equiv pr_1 + (1-p)r_2 = r_f \equiv pr_f + (1-p)r_f$$

is equivalent to $p(r_1 - r_f) = (1-p)(r_f - r_2)$, which implies $X = 1$ and $v^* = 0$. An increase in p will clearly lead to an increase in X. Then consider the derivative of v^* with respect to X:

$$\frac{\partial v^*}{\partial X} = W(1+r_f)\frac{r_1 - r_f + X(r_f - r_2) - (X-1)(r_f - r_2)}{[r_1 - r_f + X(r_f - r_2)]^2}.$$

Since the numerator can be rewritten as $r_1 - r_f + Xr_f - Xr_2 - Xr_f + Xr_2 + r_f - r_2$, which is equal to $r_1 - r_2$, the derivative is positive. A higher probability for the best outcome, and correspondingly lower probability for the worst, makes the risky asset more attractive, also for someone who is risk averse. More will be invested in the risky asset.

(e)

Based on the assumptions in (c), and with $v^* > 0$ as a starting point, what is the effect on v^* of an increase in the individual's risk aversion? Give an economic interpretation of this effect.

Answer

Increased risk aversion means an increase in $1 - b$, which will clearly lead to a decrease in X. The derivative of v with respect to X is positive, so there will also be a decrease in v^*. This is natural, since there is a trade-off here between higher expected return (when $E(\tilde{r}) > r_f$) and more risky return. Higher risk aversion will lead to a different choice when faced with this trade-off.

Problem 2

(a)

Consider a firm that can make a real investment I at time $t = 0$. If undertaken, the investment starts a project, which will have a cash flow at time $t = 1$ equal to PQ. Here, P is a product price in a competitive market, uncertain as seen from $t = 0$, while Q is a quantity, known with certainty from $t = 0$. In part (a), the firm has no other activity. Discuss what is the value of the firm, based on the Capital Asset Pricing Model. You can disregard time periods after $t = 1$. You can assume that the firm does not borrow and that there are no taxes to be paid. Please remember to give a short definition of each variable you introduce.

Answer

From the CAPM follows a valuation function, a real-valued function of the stochastic variable P, which gives the market value at $t = 0$ of a claim to one unit of the product. The function is

$$V(P) = \frac{1}{1 + r_f} \left[E(P) - \lambda \operatorname{cov}(P, r_m) \right],$$

where

$$\lambda \equiv \frac{E(r_m) - r_f}{\text{var}(r_m)}$$

Here, r_f is the risk-free interest rate, and r_m is the rate of return on the market portfolio. (This is in section 7.3 in Danthine and Donaldson and in p. 7 of lecture 9 Sept. 2013.)

Based on this function, the value of a claim to PQ is $V(P)Q$. The net value of the firm, if the investment is undertaken, is $V(P)Q - I$. Since there is no obligation to undertake the project, the value of the firm is equal to the value of the opportunity to undertake the investment. Since it cannot be undertaken later, the only question is to undertake it now or never, and it will be optimal to invest if the net value is positive. Thus the value of the firm is $\max(0, V(P)Q - I)$.

(b)

Assume now that the firm discovers another investment opportunity, with investment cost J at $t = 0$ and cash flow ZY at $t = 1$. Again, Z is an uncertain price and Y is a certain quantity, if the investment is undertaken.

What is the value of the firm when it has both opportunities? Derive a formula and discuss whether it depends on P and Z being stochastically independent.

Answer

There is a similar value $\max(0, V(Z)Y - J)$ of having the second investment opportunity. The value of having both opportunities is simply the sum, $\max(0, V(P)Q - I) + \max(0, V(Z)Y - J)$. In a CAPM economy, there is value additivity, i.e., the two projects can be valued separately. Since there is no synergy, i.e., the cash flows of the two projects are not affected by the fact that one firm undertakes both, the values are not affected by this fact. This also holds for the decision whether or not to undertake a project. The decision does not depend on whether the other project is undertaken.

There is no use here for information about whether P and Z are stochastically independent. The relevant information is the covariance of each of these with r_m.

(c)

How will the beta of shares in the firm in part (b) (call this β_b) relate to the beta of the shares in the firm in part (a) (call this β_a) and the beta of the shares in a (hypothetical) firm with only the second of the two investment opportunities (call this β_c)? How does β_b depend on I and J?

Answer

This is covered in p. 15 of the lecture of 9 September 2013, see separate page below.

The question makes little sense if no project, or only one project, is started. In those cases, one or both monoproduct firms will have shares with zero value and betas are not well defined.

If both projects are started, β_b will be a weighted average of the two other betas. The beta of a monoproduct firm is defined as

$$\beta_a = \frac{\text{cov}(\frac{P}{V(P)}, r_m)}{\text{var}(r_m)},$$

and similarly for β_c.

If the two weights are denoted w and $1 - w$, they are given by

$$w \equiv \frac{V(P)Q}{V(P)Q + V(Z)Y},$$

i.e., they are value weights, the ratios of valuations of one part of the cash flow to the valuation of the total.

We can assume that in equilibrium, a claim to one unit of a good next period will have a positive value, so both $V(P)$ and $V(Z)$ are positive. This implies that the weights are both in the interval $[0, 1]$.

Then, as shown in the lecture notes, the firms beta is $\beta_b = w\beta_a + (1-w)\beta_c$, between (i.e., a convex combination of) the two betas of the two monoproduct firms given that both projects are started.

The betas of these firms do not depend on I and J. There are no financial assets in the firm, and no financial liabilities. The way the firms are described, the owners may sell shares at the gross values (not net of investment costs) given (i.e., $V(P)Q$ and similar for the hypothetical other firm, and the sum for the combined firm) after having undertaken the investments, so that the owners' net cash inflow will be the net values described. Thus, I and J only affect the net values, but not the future cash flows, and thus not the betas.

Problem 3

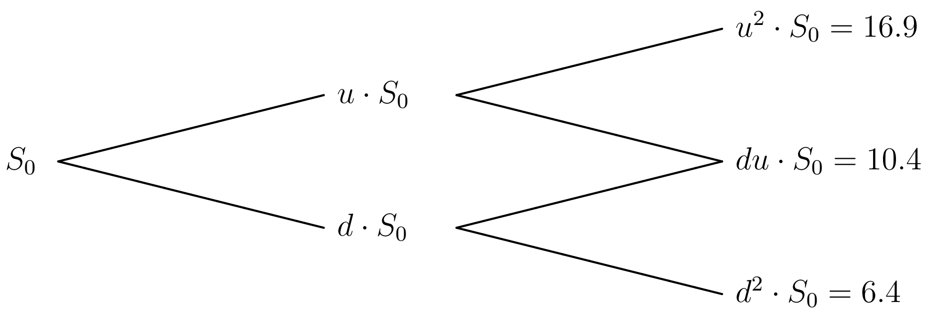
Since calculators are not allowed for this exam, there is a table at the end which provides some calculations that may or may not be useful for answering the questions that follow. In order not to make the correct answers too obvious, some useless numbers are included in the table.

(a)

Assume that the binomial share and option model is valid: Consider a share which for sure does not pay any dividend in the periods we focus on. All agents know that if the share price at time t is S_t, then the share prices at $t+1$ will be uS_t or dS_t. Consider a European call option with two periods left to maturity, when the share price today is $S_0 = 10$, the exercise price of the option is $K = 8$, the one-period interest rate factor is $e^r = 1.1$, the factor u is 1.3, and the factor d is 0.8. The probability of an upward movement is $\Pr(S_t/S_{t-1} = u) = 0.8$. Find the value of the call option at time $t = 0$.

Answer

The tree which describes the possible outcomes of the share value looks like:



The corresponding tree for the option looks like:

$$\begin{tikzcd}
 & & c_{uu} = \max(0, u^2S - K) = 8.9 \\
 & c_u \arrow[ur, dash] \arrow[dr, dash] & \\
c_0 \arrow[ur, dash] \arrow[dr, dash] & & c_{du} = \max(0, duS - K) = 2.4 \\
 & c_d \arrow[ur, dash] \arrow[dr, dash] & \\
 & & c_{dd} = \max(0, d^2S - K) = 0
\end{tikzcd}$$

The option value can be found using "risk neutral" valuation, based on an artificial probability for upward moves in the tree equal to

$$p = \frac{e^r - d}{u - d} = 0.6.$$

Using this probability, we need the expected option value at time $t = 2$, discounted back to the present using the risk free interest rate. Since the risk free rate is given in the form of a factor e^r, we use this for discounting also. The formula is thus

$$c_0 = \frac{p^2 c_{uu} + 2p(1-p)c_{du} + (1-p)^2 c_{dd}}{e^{2r}}.$$

This method can be used without any further justification. However, there is an alternative method with the construction of a replicating portfolio strategy. During the lectures, that method was used to justify the "risk neutral method". The replicating portfolio strategy method may also be used directly.

With $c_{dd} = 0$, there are two terms left in the numerator, and we find

$$c_0 = \frac{0.36 \cdot 8.9 + 0.48 \cdot 2.4}{1.1^2} = 3.6.$$

It is a mistake to use the actual probability for an upward movement, given as 0.8. Under the assumptions of this model, that probability does not influence the option value at all, except that the same states of the world must have strictly positive probabilities both under the artificial "risk neutral" probabilities and under the actual probabilities.

(b)

Another call option on the same share, with expiration at $t = 2$, has a (theoretical) value at $t = 0$ equal to 1.8. What is the exercise price of this option?

Answer

The general formula for c_0 may be used, but this time, we do not know how many of the option values at different nodes at $t = 2$ are equal to zero. The equation we want to solve, includes three expressions of the $\max(0, \cdot)$ type. We need linear variants of these equations in order to solve them. We need to consider three possible cases, (i) $duS_0 < K < u^2S_0$, (ii) $d^2S_0 < K < duS_0$, and (iii) $K < d^2S_0$. (The possibility that $16.9 = u^2S_0 < K$ can be neglected, since such a call option would clearly have no value, so this is not a solution.)

For case (i), we can try to solve

$$c_0 = \frac{0.36 \cdot (16.9 - K)}{1.1^2} = 1.8.$$

This has the solution $K = 10.85$, which satisfies the inequality, i.e., it is a valid solution.

It can be shown that this is a sufficient solution method, i.e., when there is a solution, it is the only one. But this has not been shown in the course, so it is reasonable to try the other alternatives as well.

For case (ii), we try to solve

$$c_0 = \frac{0.36 \cdot (16.9 - K) + 0.48 \cdot (10.4 - K)}{1.1^2} = 1.8.$$

This has the solution $K \approx 10.5929$, which violates the inequalities that defined case (ii). Thus, this is not a valid solution.

For case (iii), we try to solve

$$c_0 = \frac{0.36 \cdot (16.9 - K) + 0.48 \cdot (10.4 - K) + 0.16 \cdot (6.4 - K)}{1.1^2} = 1.8.$$

This has the solution $K = 9.922$, which violates the inequalities that defined case (iii). Thus, this is not a valid solution.

The conclusion is thus $K = 10.85$.

Table of calculations; some of these may be useful

$8.898/0.84$	$\approx$	10.5929	$0.48 \cdot 2.4$	$=$	1.152
$0.36 \cdot 16.9$	$=$	6.084	$0.16 \cdot 6.4$	$=$	1.024
$0.48 \cdot 10.4$	$=$	4.992	$1.092/1.21$	$\approx$	0.9025
$6.016/1.21$	$\approx$	4.9719	$0.6 \cdot 0.6$	$=$	0.36
$3.6 \cdot 1.21$	$=$	4.356	$0.55 \cdot 0.55$	$=$	0.3025
$2.814/0.84$	$=$	3.35	$0.55 \cdot 0.45$	$=$	0.2475
$0.36 \cdot 8.9$	$=$	3.204	$0.6 \cdot 0.4$	$=$	0.24
$1.8 \cdot 1.21$	$=$	2.178	$0.45 \cdot 0.45$	$=$	0.2025
$1.1 \cdot 1.1$	$=$	1.21	$0.4 \cdot 0.4$	$=$	0.16