# Exam ECON4510 spring 2015

## Guidance for grading; see indented paragraphs

### Non-indented paragraphs give the original exam text

References to curriculum are

- Danthine J.-P. and J.B. Donaldson, *Intermediate Financial Theory,* 3rd ed., Amsterdam 2015.
- Hull, J.C., *Options, Futures, and Other Derivatives,* 9th ed., Boston 2015.

References to lectures are

`http://www.uio.no/studier/emner/sv/oekonomi/ECON4510/v15/`

The exam consists of three problems. They count equally. Start by reading through the whole exam, and make sure that you allocate time to answering questions you find easy. You can get a good grade even if there are parts of problems that you do not have time to solve.

You are warned that there may be information given which is not necessary to answer the problems.

Since calculators are not allowed for this exam, there is a table at the end which provides some calculations that may or may not be useful for answering the questions that follow. In order not to make the correct answers too obvious, some useless numbers are included in the table.

# Problem 1

## (a)

Consider the function $U(W) = cW^2 + bW + a$, with $a, b$ and $c$ being constant, real numbers. You can assume that for our purpose, the function is defined only for $W \geq 0$. Discuss under what conditions this function can be a meaningful utility function for someone who is risk averse and maximizes expected utility. Sketch the graph of the function. What is the measure of absolute risk aversion, $R_A(W)$, for this utility function? Is $R_A(W)$ constant, increasing, and/or decreasing? What can you say about the advantage(s) and drawback(s) of using this function as a utility function to describe risk averse maximization of expected utility?

This is covered in D&D, sect. 4.2 and app. 6.1, and lecture 28 Jan 2015, p. 27.

We need $U'(W) = 2cW + b > 0$ and $U''(W) = 2c < 0$. Requires $c < 0, b > 0$, and a restriction that $W < -b/(2c)$. This implies a restriction on the probability distribution of $W$, or any $W$ that may be chosen, so that $W < -b/(2c)$ holds with full certainty. In that case the utility function is meaningful. The graph is an upside-down parabola with a maximum point for $W = -b/(2c) = |b/(2c)|$.

We find $R_A(W) = -U''(W)/U'(W) = -2c/(2cW + b)$. The first derivative is $dR_A(W)/dW = 4c^2/(2cW + b)^2$, which is positive. This implies increasing absolute risk aversion.

The restriction on possible $W$ values is a drawback for the use of the function. If some parameters $c, b$ are found to describe a person's choices in one situation, they can perhaps not be used if $W$ can take on much higher values in another situation.

Increasing absolute risk aversion is also a drawback. Most studies and most intuition points to decreasing absolute risk aversion as more appropriate.

The advantage of the quadratic function is that it facilitates calculations, compared with many alternatives.

## (b)

Assume now that conditions are satisfied so that the function defined in (a) can be meaningfully used. Show that a person with this utility function, planning for one future period with risky wealth, will care only about the mean and variance of wealth, $E(W)$ and $\text{var}(W)$.

From the lecture,

$$\begin{aligned} E[U(\tilde{W})] &= cE(\tilde{W}^2) + bE(\tilde{W}) + a \\ &= c\{E(\tilde{W}^2) - [E(\tilde{W})]^2\} + c[E(\tilde{W})]^2 + bE(\tilde{W}) + a \\ &= c\,\text{var}(\tilde{W}) + c[E(\tilde{W})]^2 + bE(\tilde{W}) + a, \end{aligned}$$

which is a function only of mean and variance of $\tilde{W}$.

## (c)

For this part, you can rely on the following property of a uniform distribution: If $W$ is uniformly distributed on the interval $[0, 6]$, then $E(W) = \text{var}(W) = 3$. Discuss the following statement: "Any person with the quadratic utility function defined in (a) will be indifferent between the following two alternatives: (i) a risky future wealth that is lognormally distributed with $E(W) = 3$ and $\text{var}(W) = 3$, and (ii) a risky future wealth that is uniformly distributed on the interval $[0, 6]$."

> For this question we assume that there is only one future period, and that the "risky future wealth" gives the complete consumption budget for the individual in that period. The two alternatives have the same mean and the same variance, which suggests that the person should be indifferent between them. This follows from the fact that mean and variance are the only two characteristics he/she cares about.
>
> However, there is no upper bound on the lognormal distribution, so the utility function cannot be applied. Regardless of parameters, there will be a positive probability for a range of outcomes greater than $-b/(2c)$. The students should be able to point this out.

## (d)

Discuss the following statement: "Any person with the quadratic utility function defined in (a) will be indifferent between the following two alternatives: (i) a risky future wealth that is lognormally distributed with $E(W) = 3$ and $\text{var}(W) = 3$, and (ii) a risky future wealth that is normally distributed with $E(W) = 3$ and $\text{var}(W) = 3$."

> This can be answered as part (c), except that there is now also a probability of negative wealth in one of the alternatives. The utility function is not defined for negative wealth. It is not clear what $W < 0$ would mean during the final (only) period of the model, if the person is supposed to consume the wealth. This means that the restriction $W \geq 0$ has an economic justification.

# Problem 2

Consider an economy described by the standard Capital Asset Pricing Model (CAPM) between two points in time, $t = 0$ and $t = 1$. Consider a potential investment $I$ at $t = 0$ to start a firm to produce a known homogeneous output $Q$ to be sold an uncertain price $\tilde{P}$ at $t = 1$. You can assume that immediately after the investment, shares in the firm will be traded in the stock market, and, for simplicity, that the firm does not pay taxes.

## (a)

Show how the CAPM $\beta$ of the shares of the firm, as described in the introduction, will be related to the covariance between the output price and the rate of return on the market portfolio, $\tilde{r}_m$. What will be the relation between $\beta$ and the amount of real investment, $I$? Give a brief verbal interpretation.

This is covered in lecture 11 Feb 2015, pp. 16–18, and 25 Feb, p. 4 (pages included below), and to some extent in D&D, sect. 8.3.

In the lecture notes, 11 Feb 2015, p. 17, a function $V()$ is defined,

$$V(\tilde{p}_{I1}) = \frac{1}{1 + r_f} \left[ E(\tilde{p}_{I1}) - \lambda \operatorname{cov}(\tilde{p}_{I1}, \tilde{r}_M) \right],$$

which transforms a scalar stochastic variable to a real number. This is a CAPM based one-period valuation function, and is used repeatedly in the lecture notes. The expression on the right-hand side is in p. 216 of D&D.

The shares are claims on $\tilde{P}Q$, so the total value of the shares is $V(\tilde{P}Q) = QV(\tilde{P})$ (This factoring out, of $Q$, is a special case of the factoring out done in p. 4 of the lecture of 25 Feb. In the lecture, quantity was uncertain, but independent, while here, it is certain.) The CAPM $\beta$ is thus equal to

$$\frac{1}{\sigma_m^2} \operatorname{cov} \left[ \frac{\tilde{P}}{V(\tilde{P})}, \tilde{r}_m \right] = \frac{1}{V(\tilde{P})\sigma_m^2} \operatorname{cov}(\tilde{P}, \tilde{r}_m).$$

This is the basic satisfactory answer. But a more advanced answer will observe that the covariance is also hidden in $V(\tilde{P})$, which leads to the expression at the bottom of p. 18 in the lecture of 11 Feb,

$$= \frac{\operatorname{cov}(\tilde{P}, \tilde{r}_m)}{\frac{\sigma_m^2}{1+r_f}[E(\tilde{P}) - \lambda \operatorname{cov}(\tilde{P}, \tilde{r}_m)]} = \frac{1 + r_f}{\sigma_m^2 \frac{E(\tilde{P})}{\operatorname{cov}(\tilde{P}, \tilde{r}_m)} - E(\tilde{r}_m) + r_f},$$

where $\lambda$ is defined by

$$\lambda \equiv \frac{E(\tilde{r}_m) - r_f}{\sigma_m^2}.$$

In none of these expressions is there any reference to $I$. The $\beta$ will not depend on $I$. If there is any net value to the investment opportunity, this is $QV(\tilde{P}) - I$, and will be cashed in at $t = 0$ by the original investors. The original amount of real investment will thus not affect the returns to those who subsequently trade the shares in the market.

## (b)

Assume now that a fraction $(1 - a) \in (0, 1)$ of the investment, $I$, will be financed by debt. Assume for simplicity that the debt can and will be paid back with interest with full certainty at $t = 1$. Under these assumptions, explain how the decision by the original investors, whether to invest or not, is related to the fraction $(1 - a)$ or $a$. Give a brief verbal interpretation.

This is covered in lecture 25 Feb, pp. 7–9 (pages included below). The result is mentioned in D&D, p. 45, but the main treatment in D&D is in chapter 17, not on the reading list.

The lecture notes show that the decision does not depend on the fraction financed by debt. This follows from value additivity: The value of receiving a loan and paying it back is added to the value of the firm. This net value of the loan, to be added, is clearly zero.

## (c)

Explain how the $\beta$ of shares in the firm, when it has the obligation to pay back debt with interest as in part (b), will be related to the the fraction $(1 - a)$ or $a$. Give a brief verbal interpretation.

The lecture notes (p. 9 in particular) show that $\beta$ will be increasing in the debt fraction (or, to be precise, inversely proportional to the equity fraction $a$) when the project is exactly marginal. This can be explained as follows: The debt service at $t = 1$ is a fixed obligation to be paid out of $\tilde{P}Q$. A higher debt service thus makes the return more risky.

This is a basic satisfactory answer.

An additional point (see the notes, p. 9, bottom): When debt is a constant proportion of $I$, a positive net value (i.e., $QV(\tilde{P}) > I$) will reduce $\beta$ towards its value in the absence of debt.

# Problem 3

## (a)

Explain in words why a put option value is decreasing in the value of the underlying asset, and why the value of an American put option is greater than or equal to the value of an otherwise identical European put option. Can you prove this second statement by absence of arbitrage?

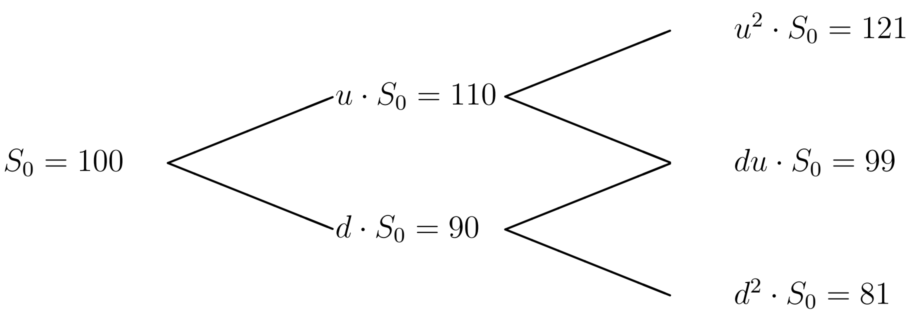
At the expiration date, $T$, the put option has the value $P_T = \max(0, K - S_T)$, where $K$ is the exercise price and $S_T$ is the value of the underlying asset. This is clearly a decreasing function of $S_T$, although not strictly decreasing. The graph in a $S_T, P_T$ diagram is a line with one kink, first with slope $-1$ from $(0, K)$ to $(K, 0)$, then horizontal at zero for $S_T > K$. At an earlier date, if the option is European, the option is just a claim to that decreasing function at $T$. Since an earlier $S_t$ is the price of a claim to $S_T$, the put option will also be decreasing in $S_t$. There is an additional value if the option is American, since this option gives the owner additional possibilities of early exercise. If the market value of the American option had been lower, there would be an arbitrage possibility, write a European option and buy an American, cash in the positive difference now, wait until $T$, at which time there would be no net obligation.

## (b)

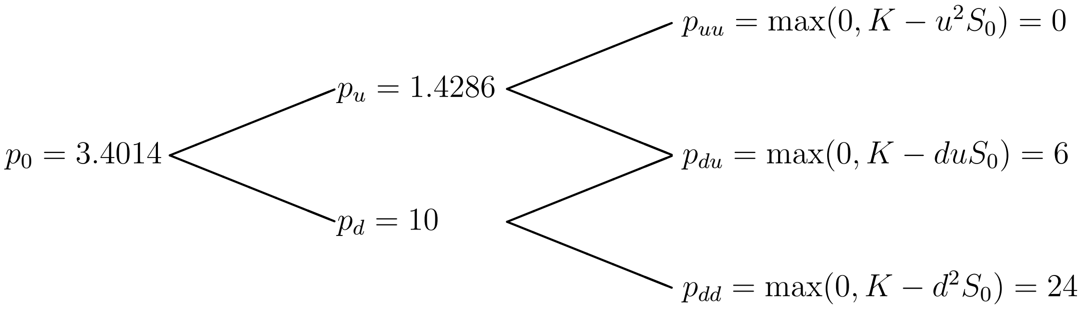
Consider a put option with expiration two periods into the future ($T = 2$), on a stock which for sure does not pay dividends between now (time zero) and then ($T$). The stock price now is $S_0 = 100$. The exercise price of the put option is $K = 105$. It is known that during each period, there is a probability $p^* = 0.9$ that the stock will increase in value by 10 percent, and a probability $1 - p^*$ that the stock will decrease in value by 10 percent. There exist risk free bonds with continuous compounding such that a risk free interest rate $r$ implies an interest factor $e^r = 1.05$ per period.

What is the value of the put option at time zero in case it is a European option?

> This is covered in Hull, sects. 13.4 and 13.5, and exercise 13.22 (which has been discussed).
>
> The only thing we need $p^*$ for, is to observe that the probabilities of the two outcomes mentioned add up to unity, i.e., there are only these two outcomes each period that have strictly positive probabilities. This means we have a binomial tree for the share price.
>
> Apart from this, $p^*$ is useless, cf. the warning in the introduction that there may be useless information.




The corresponding tree for the European option looks like:




The values at the nodes at $T = 2$ follow directly from the definition of a put option. The option value at $t = 0$ could be found using a two-period formula, but we need the intermediate values at $t = 1$ in part (c) below.

In the calculations, we need the risk-free probability of an upward movement in the tree, $p = (e^r - d)/(u - d) = 0.75$. We follow Hull's notation, so that lower-case $p$ without subscript is this probability, while lower-case $p$ with subscripts are values of European put options. Upper-case $P$ with subscripts are American put values.

The option value at node $u$ at $t = 1$ is

$$p_u = e^{-r}(pp_{uu} + (1-p)p_{ud}) = \frac{1}{1.05}(0.75 \cdot 0 + 0.25 \cdot 6) = \frac{1.5}{1.05} \approx 1.4286.$$

Similarly for node $d$,

$$p_d = e^{-r}(pp_{ud} + (1-p)p_{dd}) = \frac{1}{1.05}(0.75 \cdot 6 + 0.25 \cdot 24) = \frac{10.5}{1.05} = 10.$$

The option value that is asked for, at node 0, is

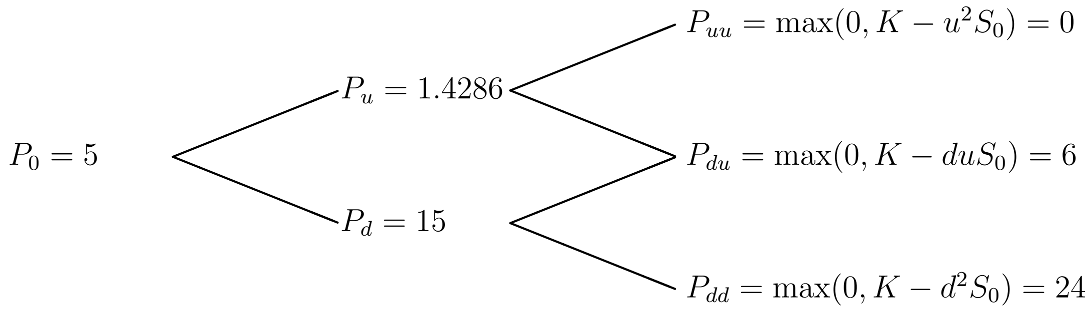
$$p_0 = e^{-r}(pp_u + (1-p)p_d) \approx \frac{1}{1.05}(0.75 \cdot 1.4286 + 0.25 \cdot 10) \approx \frac{3.5714}{1.05} \approx 3.4014.$$

## (c)

What is the value at time zero of the put option mentioned in part (b) in case it is an American option?

This is covered in Hull.

The corresponding tree for the American option looks like:




There is a possibility of early exercise, which can be evaluated at each node before $T$ by comparing the option value if kept alive with the exercise value. At $t = 1$, the option value if kept alive until $T$ is equal to the European values at each of the two nodes. The value if exercised is $K - uS_0 = 105 - 110 = -5 < 0$ at the upper node, which is clearly a bad alternative. At the lower node, the value if exercised is $K - dS_0 = 105 - 90 = 15$. This is greater than the value if kept alive, which is 10. Thus early exercise is optimal at this node.

The value at $t = 0$ if kept alive takes the early exercise at node $d$ at $t = 1$ into account. The value if kept alive is thus

$$P_{0,\text{alive}} = e^{-r}(pP_u + (1-p)P_d) \approx \frac{1}{1.05}(0.75 \cdot 1.4286 + 0.25 \cdot 15) \approx \frac{4.8214}{1.05} \approx 4.5918.$$

This must be compared with the exercise value at $t = 0$, which is $K - S_0 = 105 - 100 = 5$. Early exercise is optimal, and we conclude that $P_0 = 5$.

**Table of calculations; some of these may be useful**

| | | | | | |
|---|---|---|---|---|---|
| $0.9 \cdot 16$ | $=$ | $14.4$ | $4.8214/1.05$ | $\approx$ | $4.5918$ |
| $14.4/1.05$ | $\approx$ | $13.7143$ | $3.5714/1.05$ | $\approx$ | $3.4014$ |
| $0.9 \cdot 13.7143$ | $\approx$ | $12.3429$ | $2.0143/1.05$ | $\approx$ | $1.9184$ |
| $12.3429/1.05$ | $\approx$ | $11.7551$ | $1.5/1.05$ | $\approx$ | $1.4286$ |
| $12/1.05$ | $\approx$ | $11.4286$ | $1.2571/1.05$ | $\approx$ | $1.1973$ |
| $0.75 \cdot 11.4286$ | $\approx$ | $8.5714$ | $1.4286 \cdot 0.75$ | $\approx$ | $1.0714$ |
| $8.5714/1.05$ | $\approx$ | $8.1633$ | $0.6/1.05$ | $\approx$ | $0.5714$ |
| $7.8/1.05$ | $\approx$ | $7.4286$ | $0.9 \cdot 0.5714$ | $\approx$ | $0.5143$ |

## Investment project's rate of return

- Consider (potential) real investment project:
  - Outlay $I$ at $t = 0$.
  - Revenue $\tilde{p}_{I1}$ at $t = 1$.
- Project value? Should project be undertaken?
- Assume 100% equity financed. (Assume separate firm?)
- Project's rate of return is $(\tilde{p}_{I1} - I)/I$.
- If use of restricted technology or resources: No reason for SML equation to hold for this rate of return.
- May earn "above-normal" expected rate of return.



## Investment project's rate of return, contd.

- However: Valuation of $\tilde{p}_{I1}$ possible:

$$p_{I0} = V(\tilde{p}_{I1}) = \frac{1}{1+r_f}\left[E(\tilde{p}_{I1}) - \lambda \operatorname{cov}(\tilde{p}_{I1}, \tilde{r}_M)\right]$$

  defines $p_{I0}$ *independently of* $I$.
- If $p_{I0} > I$, undertake project. Net value $p_{I0} - I$.
- If $p_{I0} < I$, drop project. Net value of opportunity is 0. Net value of *having to* undertake project is $p_{I0} - I$, negative.
- Competition and free entry $\Rightarrow$ $p_{I0} = I$ in long run, through increased supply, lower $E(\tilde{p}_{I1})$.



## Project valuation

- If claim to $\tilde{p}_{I1}$ costs (formula) $p_{I0}$, this is equilibrium price.
- Project is then on security market line (SML).
- If project available at different cost $I$: Not at SML.

The equilibrium ratio $E(\frac{\tilde{p}_{I1}}{p_{I0}})$ corresponds to $\beta_k$ in

$$E(\frac{\tilde{p}_{I1}}{p_{I0}}) = 1 + r_f + [\mu_M - r_f]\beta_k$$

$$= 1 + r_f + [\mu_M - r_f]\operatorname{cov}(\frac{\tilde{p}_{I1}}{p_{I0}}, \tilde{r}_M)/\sigma_M^2.$$

Expressed in terms of exogenous variables (eliminating $p_{I0}$) this becomes:

$$\beta_k = \frac{(1+r_f)}{\sigma_M^2 \frac{E(\tilde{p}_{I1})}{\operatorname{cov}(\tilde{p}_{I1}, \tilde{r}_M)} - \mu_M + r_f},$$

independent of $I$. Only if $I = p_{I0} = V(\tilde{p}_{I1})$, will the project rate of return $\tilde{p}_{I1}/I - 1$ satisfy the CAPM equation with $\beta_k$.



## Example, contd.

$\tilde{p}_{I1} = \tilde{P}_i\tilde{X}_i + \tilde{P}_j\tilde{X}_j - \tilde{P}_k\tilde{X}_k.$

*Valuation of product of stochastic variables*

Quantity uncertainty often local, technical, meteorological. May simplify valuation of $\tilde{P}\tilde{X}$ expressions if assume: Each $\tilde{X}_h$ $(h = i, j, k)$ is stochastically independent of $(\tilde{P}_h, \tilde{r}_M)$. Then: $E(\tilde{P}\tilde{X}) = E(\tilde{P})E(\tilde{X})$ and

$$\operatorname{cov}(\tilde{P}\tilde{X}, \tilde{r}_M) = E(\tilde{P}\tilde{X}\tilde{r}_M) - E(\tilde{P}\tilde{X})E(\tilde{r}_M)$$

$$= E(\tilde{X})\left[E(\tilde{P}\tilde{r}_M) - E(\tilde{P})E(\tilde{r}_M)\right] = E(\tilde{X})\operatorname{cov}(\tilde{P}, \tilde{r}_M) \Rightarrow$$

$V(\tilde{P}\tilde{X}) = E(\tilde{X})V(\tilde{P})$, quantity uncertainty irrelevant.

## Application: Borrowing to finance part of investment

- Consider a firm that raises $aI$ from shareholders and $(1-a)I$ as a loan to finance real investment $I$, with $a \in (0, 1)$.
- Assume investment results in a cash flow $\tilde{p}_I$ one period later.
- Loan will be paid back with interest, a total of $(1-a)(1+r_f)I$.
- For simplicity: Assume now that payback will happen with certainty, irrespective of the outcome of $\tilde{p}_I$.
- This is called a risk free loan, which is not the typical case in practice.
- (Typical case: There is some chance of default (konkurs), in which case the loan is not paid back in full. Ignore this now.)
  - The loan may be risk free if $\Pr(\tilde{p}_I > (1-a)(1+r_f)I) = 1$, which may occur because the uncertainty about the outcome of $\tilde{p}_I$ is not too big, and $(1-a)$ is not too big.
  - The loan could also be risk free because the shareholders somehow guarantee to pay back, by putting up collateral, in addition to $\tilde{p}_I$.
  - We assume the loan must be risk free for the lender to be satisfied with $r_f$ as the promised interest rate.



## Borrowing: Consequences for investment decision

- Value today of shareholder's claim is:

$$p_0 = V(\tilde{p}_I - (1-a)(1+r_f)I) = V(\tilde{p}_I) - V((1-a)I(1+r_f))$$
$$= \frac{1}{1+r_f}[E(\tilde{p}_I) - \lambda \operatorname{cov}(\tilde{p}_I, \tilde{r}_M)] - (1-a)I.$$

- The decision to invest or not depends on whether this is $> aI$, which can be simplified to:

$$\frac{1}{1+r_f}[E(\tilde{p}_I) - \lambda \operatorname{cov}(\tilde{p}_I, \tilde{r}_M)] > I.$$

- There is no $a$ in this inequality.
- The investment decision does not depend on the loan financing.



## Borrowing: Consequences for beta of shares

- When correctly priced at $p_0$, the beta of the shares is

$$\frac{1}{\sigma_M^2} \operatorname{cov}\left[\frac{\tilde{p}_I - (1-a)(1+r_f)I}{p_0}, \tilde{r}_M\right]$$
$$= \frac{V(\tilde{p}_I)}{V(\tilde{p}_I) - (1-a)I} \cdot \frac{1}{\sigma_M^2} \operatorname{cov}\left[\frac{\tilde{p}_I}{V(\tilde{p}_I)}, \tilde{r}_M\right].$$

- When, also, project is exactly marginal, $p_0 = aI$, the beta is

$$\frac{1}{a} \cdot \frac{1}{\sigma_M^2} \operatorname{cov}\left[\frac{\tilde{p}_I}{V(\tilde{p}_I)}, \tilde{r}_M\right] = \frac{1}{a} \cdot \beta_{(\text{without borrowing})}$$

  higher, the more is borrowed, i.e., the higher is $(1-a)$ and $\frac{1}{a}$.
- For a non-marginal project, a higher $V(\tilde{p}_I)/I$ will reduce beta.