Exam ECON4510 spring 2016

Guidance for grading; see indented paragraphs

References to curriculum are

- Danthine J.-P. and J.B. Donaldson, *Intermediate Financial Theory,* 3rd ed., Amsterdam 2015.
- Hull, J.C., *Options, Futures, and Other Derivatives,* 9th ed., Boston 2015.

References to lectures are

`http://www.uio.no/studier/emner/sv/oekonomi/ECON4510/v16/index.html`

The exam consists of three problems. They count equally. Start by reading through the whole exam, and make sure that you allocate time to answering questions you find easy. You can get a good grade even if there are parts of problems that you do not have time to solve.

Problem 1

(a)

Explain what is meant by riskless arbitrage, and why we assume such arbitrage is incompatible with a market equilibrium.

Answer

(See lect. 4 Apr., pp. 7–8.)

Riskless arbitrage: A set of transactions which gives us a net gain now, and with certainty no obligation to pay out a net positive amount at any future date. If such opportunities exist, everyone would demand infinite amounts of them, not compatible with equilibrium.

(b)

Use the concepts from part (a) to prove that under some conditions, a forward price is determined by the equation $F_0 = S_0 e^{rT}$, where S_0 is the price of some asset. Be careful to explain the meaning of the variables and the conditions needed for your proof to be valid.

Answer

(See Hull, sect. 5.4, and lect. 4 Apr., pp. 2–10.)

F_0 is the forward price determined for a new forward contract at time zero, when the delivery date is T. The riskless interest rate is r with continuous compounding. (Strictly speaking, this is the interest rate over the period from 0 to T, but this has not been stressed in the course.)

The asset must be an investment asset, so that the arbitrage can be used both ways. If not an investment asset, investors will in general not be willing to hold the asset. The asset is assumed not to make payouts.

The arbitrage proof is

- If not, one could make a risk free arbitrage, buying the cheaper, selling the more expensive:
- If $F_0 e^{-rT} > S_0$; buy underlying asset, sell bonds (i.e., borrow) in amount $F_0 e^{-rT}$, sell forward contract, make net positive profit now (time zero) equal to $F_0 e^{-rT} - S_0 > 0$. At delivery date; deliver underlying asset, receive F_0, pay back loan, with no net payout and no remaining obligations.
- If $F_0 e^{-rT} < S_0$; do the opposite. This involves short-selling the underlying asset. If this is not possible (e.g., the underlying asset is gold, and no one lets you borrow gold in order to sell it immediately), then, at least, those who own the asset now have the opportunity to sell it now and earn the arbitrage profit.
- In both cases there is thus an arbitrage opportunity.

(c)

Consider a forward contract that satisfied the equation from (b) when it was entered into. What will be the value of (a long position in) the contract at some later point in time? Explain!

Answer

(See Hull, sect. 5.7, and lect. 4 Apr., p. 13.)

Assume that the "later point in time" is still before the delivery date. Below, this new date is called date 0, and the assumption is that the forward contract was entered into before date 0.

- Consider now a forward contract which was entered into some time before now (—now is date zero); the underlying asset is assumed to be an investment asset; for sure no payouts
- K is the price written into the contract, while F_0 is the equilibrium forward price now, both referring to same delivery date, T.
- S_0 and expectations of S_T will (typically) have changed since the time when K was determined.
- We can consider $K \neq F_0$ as a kind of mispricing of the contract, which means that owning the contract now has a positive or negative value.
- The value is $f = (F_0 - K)e^{-rT}$, and there is no reason to believe this is zero.
- For the case of an investment asset without payouts, $F_0 e^{-rT} = S_0$, and $f = S_0 - Ke^{-rT}$, which is the valuation of S_T minus the valuation of the obligation to pay K.

(d)

If it is revealed that the asset with price S_0, mentioned in part (b), will give a payout (e.g., a dividend) before the delivery date, what will be the effect on the forward price, if any? You can assume that before this was revealed, the market assumed that there would be no payout.

Answer

(See Hull, sect. 5.5, and lect. 4 Apr., pp. 11–12.)

When a payout is revealed, the market will realize that S_0 is no longer a claim to S_T, but to the payout and S_T. The forward contract is only a claim to S_T, however, so the relevant price of the underlying asset is not S_0, but S_0 minus the present value of the payout. If the payout is known, an arbitrage proof can be used to show this.

Problem 2

Consider the well-known function

$$c(S_0, K, r, \sigma, T) = S_0 N(d_1) - K e^{-rT} N(d_2),$$

where $N(\cdot)$ is the standard normal cumulative distribution function, and the variables d_1 and d_2 are defined by

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}},$$

and

$$d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}.$$

(a)

Explain how the five arguments of the c function are defined, and what the function is meant to give us. Explain the main conditions needed for the formula to be valid. You are not asked to give a mathematical proof that the formula holds under those conditions.

Answer

(See Hull, sect. 15.1–15.8, lect. 25 Apr. and 2 May.)

The formula gives the value at time 0 of a European call option on a stock that does not pay dividends between now and the expiration date (T) of the option. The stock has price S_0 at time

0, and the exercise price is K. The stock is supposed to follow a geometric Brownian motion with drift, with constant volatility, $\sigma = \sqrt{\mathrm{var}[\ln(S_t/S_{t-1})]}$. There is supposed to be a constant risk free interest rate, r. If there are no transaction costs, no short sale constraints, and continuous trading, the formula will hold, or there will be arbitrage opportunities. (Additionally, one could mention that there should be no taxes and full divisibility of assets. One could also mention that the absence-of-arbitrage argument will hold even if agents disagree on the expected growth rate of the stock price. But these additional points are not "main conditions," and give little extra credit.)

(b)

Explain how the function, under some conditions, may be useful to find a forward-looking estimate of σ.

Answer

(See Hull, sect. 15.11, lect. 2 May pp. 22–23.)

The volatility, σ, is the only variable that is not directly observable. It is observable in a statistical sense. Under the assumption that the model holds, and observed call option value c_{obs} will be equal to the theoretical $c(S_0, K, r, \sigma, T)$ for one value of σ, which can be seen as that forward-looking estimate of σ that is reflected in the market price. The equation cannot be solved analytically, but for any set of variables $c_{\text{obs}}, S_0, K, r, T$, the equation can be solved numerically.

(c)

When S_0 becomes very large, and the other variables are fixed, the c function approaches the value described in part (c) of Problem 1 above. Show this mathematically. Give a verbal interpretation.

Answer

(See Hull, pp. 337–338.)

When S_0 becomes very large, both d_1 and d_2 become very large (under the assumption that the other variables, K, σ, r, T, are fixed). Thus, $N(d_1)$ and $N(d_2)$ are approaching 1.0, and the call option value, according to the formula, approaches $S_0 - Ke^{-rT}$, which is exactly the value of an existing forward contract, found in part (c) of Problem 1. This is natural, since the option is now almost certain to be exercised, since the probability that $S_T < K$ is approaching zero. In the limit, the option is thus like a forward contract with K as the forward price.

Problem 3

Consider an economy described by the standard Capital Asset Pricing Model (CAPM) between two points in time, $t = 0$ and $t = 1$. Consider a potential investment I at $t = 0$ to produce a known homogeneous output Q to be sold an uncertain price $\tilde{P}$ at $t = 1$. No other factor input is necessary. The output quantity is determined with full certainty by the amount invested through the function $Q = f(I) = aI^b$, where a and b are positive constants. The valuation at $t = 0$ of a claim to one unit of output at $t = 1$ is denoted $V(\tilde{P})$, and is assumed to be strictly positive. For simplicity, assume that the firm has no other activity. In part (a) and (b) the firm does not pay taxes.

(a)

What is the first-order condition for choice of an optimal amount to invest? Explain why we may want to assume $b < 1$. Explain how and why the optimal choice does, or does not, depend on the risk aversion of the owners of the firm.

Answer

(See lect. 8 Feb., pp. 6–7, D& D, p. 216.)

The cash flow at $t = 1$ is $\tilde{P}Q = \tilde{P}aI^b$. The valuation at $t = 0$ of a claim to this is, according to the CAPM, $V(\tilde{P})aI^b$, where the function V takes as its argument a stochastic variable to be realized at $t = 1$, and returns a real number, its valuation. The function is defined by

$$V(\tilde{P}) = \frac{1}{1+r_f}\left[E(\tilde{P}) - \lambda \operatorname{cov}(\tilde{P}, \tilde{r}_m)\right],$$

where λ is defined by

$$\lambda = \frac{E(\tilde{r}_m) - r_f}{\operatorname{var}(\tilde{r}_m)},$$

r_f is the risk free interest rate, and $\tilde{r}_m$ is defined in part (b).

That $V(\tilde{P}Q) = V(\tilde{P})Q$ holds, is because a non-stochastic factor can be factored out of both the expectation and the covariance.

The firm will want to maximize the net market value,

$$\max_I \left[-I + V(\tilde{P})aI^b\right],$$

with first-order condition $-1 + abI^{b-1}V(\tilde{P})$. This can be solved for $I = \left[V(\tilde{P})ab\right]^{1/(1-b)}$.

The second-order derivative with respect to I is $ab(b-1)I^{b-2}V(\tilde{P})$. The second-order condition says this should be negative, i.e., $0 < b < 1$. Then there are decreasing returns to scale. If not, there are constant or increasing returns to scale, and there is nothing in the model described which gives a finite solution to the optimization problem. (We should also check whether the interior optimum always gives a positive net value. It does, because $f'(0)$ is infinite, so the first unit produced is always profitable, irrespective of $V(\tilde{P})$. But there is no question about this, and no answer can be expected.)

The risk aversion of any particular set of owners does not matter in this model, since it is assumed that the shares are traded in a stock market, and the owners are well diversified in the market. They are only interested in maximization of the net market value, as shown. An additional point can be made, however: When the model is taken literally, everyone in this closed economy owns shares in every firm. "The owners" are thus everyone, and their risk aversion will be reflected in $\tilde{r}_m$ and λ.

(b)

Show that the first-order condition leads to $I/Q = bV(\tilde{P})$.

> The production function implies $I/Q = I/aI^b = I^{1-b}/a$. When the expression for optimal I is plugged in, this yields $I/Q = V(\tilde{P})ab/a = V(\tilde{P})b$.

(c)

Show how the CAPM β of the shares of the firm, immediately after investment, will be related to the covariance (assumed to be positive) between the output price and the rate of return on the market portfolio, $\tilde{r}_m$. Will there be a relation between β and the parameter b of the production function? Give a brief verbal interpretation.

Answer

> The shares are claims to $\tilde{P}Q$, and the return (i.e., one-plus-the-rate-of-return) is given as $\tilde{P}Q/V(\tilde{P})Q = \tilde{P}/V(\tilde{P})$. The β is defined as $\beta_P = \operatorname{cov}(\tilde{P}/V(\tilde{P}), \tilde{r}_m)/\operatorname{var}(\tilde{r}_m)$, in which the covariance mentioned in the problem text can be factored out,
>
> $$\beta_P = \frac{1}{V(\tilde{P})\operatorname{var}(\tilde{r}_m)} \operatorname{cov}(\tilde{P}, \tilde{r}_m).$$
>
> This does not depend on b. Since the claims are pure claims on output, and the β refers to a point in time after investment has been made, there is no relation to the sunk cost of investment.

(d)

Assume now that the firm pays a tax in $t = 1$ (but not at $t = 0$). The tax payment is $\tau \cdot (\tilde{P}Q - cI)$, where $\tau \in (0, 1)$ is a tax rate, and $c = 1 + r_f$. The cash flow to the firm at $t = 1$ is thus $\tilde{P}Q(1 - \tau) + \tau cI$. For simplicity, assume that the firm will earn the tax value of the deduction, τcI, at $t = 1$ even when $\tilde{P}$ has low outcomes, so that τcI can be considered as a risk free cash flow.

Show in this case how the CAPM β of the shares of the firm, immediately after investment, will be related to the covariance between the output price and the rate of return on the market portfolio, $\tilde{r}_m$. Will there in this case be a relation between β and the parameter b of the production function? Give a brief verbal interpretation.

Answer

(See lect. 8 Feb. p. 15, on value-weighted average.)

The shares are claims to $\tilde{P}Q(1-\tau)+\tau cI$. The valuation of this is

$$V(\tilde{P})Q(1-\tau)+\tau cI/(1+r_f) = V(\tilde{P})Q(1-\tau)+\tau I.$$

The β will be a value-weighted average of the βs of the elements of the cash flow. The risk free element has a β of zero. The value weight of the risky element is

$$\frac{V(\tilde{P})Q(1-\tau)}{V(\tilde{P})Q(1-\tau)+\tau I} = \frac{V(\tilde{P})(1-\tau)}{V(\tilde{P})(1-\tau)+\tau I/Q}$$

which is decreasing in I/Q, and thus in b, based on the formula found in part (b). The β of the shares is equal to this weight multiplied by β_P from part (c),

$$\frac{V(\tilde{P})(1-\tau)}{V(\tilde{P})(1-\tau)+\tau I/Q}\beta_P,$$

so this is also decreasing in b.

More precisely, this relies on the same solution for I being optimal, which is the case because the value to be maximized is $(1-\tau)[V(\tilde{P})Q-I]$, so that the maximum is independent of τ.

The interpretation is that b closer to 0 means higher gross value, $V(\tilde{P}Q)$, relative to I, so that the shares are closer to a claim on $\tilde{P}Q$, with a higher β. On the other hand, if b is close to unity, I is close to $V(\tilde{P})Q$, so there is little surplus, and a larger fraction of next period's cash flow consists of the risk free value of the tax deduction. This means the β is closer to zero.