Exam in econ4510

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YOU MAY ANSWER IN ENGLISH OR NORWEGIAN.

Some advice: Start by reading through the whole exam, and make sure that you allocate time to answering questions you find easy. You can get a good grade even if there are parts of problems that you do not have time to solve. It is better to try to do something on each question than to get bogged down with one question. If you find you are spending too much time on one question, stop working on it and plan to get back to it if you have time at the end. Make sure you state any assumptions you make.

1. **CAPM [35%]**: Assume that all investors have linear-quadratic preferences and suppose they can invest their wealth in a set of risky assets. Moreover, the returns on the assets are imperfectly correlated and the expected return and variance differs across assets.

   (a) [5%] Suppose there is no risk free asset. Illustrate, using a diagram, the set of portfolios that investors will choose in equilibrium.

   ANSWER: Draw a diagram with risk (standard deviation of return) on the x-axis and expected return on the y-axis. Draw a hyperbola representing the efficient set, i.e., the upper part of the hyperbola is the "efficient set", i.e., the set of optimal portfolios. All assets should be to the right of the hyperbola and the market portolio should be on the efficient set.

   (b) [5%] Explain how the optimal portfolios change when a risk free asset is introduced

   ANSWER: the efficient set now becomes a linear combination of the risk-free rate and the market portfolio (two-fund separation), i.e., the Cecurity Market Line

   (c) [10%] Suppose a risky asset i has an expected return equal to the risk free rate $(E(r_i) = r_f)$ and the same standard deviation as the return on the market portfolio.

      i. Explain why all individuals want to hold this asset in equilibrium even though it offers a low equilibrium return.

      ANSWER: In equilibrium, the market portfolio is efficient and with two-fund separation all investors will hold

the market portfolio (in combination with bonds). Asset i is part of the market portfolio and the price of the asset must threfore be such that investors are willing to hold it.

ii. What must the covariance be between r_i and the return on the market portfolio in equilibrium?
ANSWER: The equilibrium will be charactedized by the CAPM equation, $E(r_i) = r_f + \beta_i (E(r_M) - r_f)$, where $\beta_i = cov(r_i, r_M)/var(r_M)$. Therefore, $cov(r_i, r_M) = 0$ implies $\beta_i = 0$ and $E(r_i) = r_f$.

(d) [15%] Empirical testing of CAPM:

i. Explain how CAPM can be tested empirically.
ANSWER: The cental theoretical implication of CAPM is that all assets will be located at the Security Market Line (SML), i.e., that the expected return is given by $E(r_i) = r_f + \beta_i (E(r_M) - r_f)$. If one can find an asset or a portfolio strategy which systematically locates an asset away from the SML, i.e., $E(r_i) \neq r_f + \beta_i (E(r_M) - r_f)$, then this would serve as an empirical rejection of CAPM.

ii. Review some empirical evidence that CAPM should be rejected.
ANSWER: Assume that the βs of all stocks are constant over (some possibly limited) time interval. Use part of this interval to estimate β_i and use the rest to test if portfolio i on average delivers the return associated with the SML. A large literature in finance has shown that thre are strategies which can "beat" CAPM. These strategies are often referred to as "factor strategies", and include for example the purchase of value stocks (HML), small cap stocks (SML), stocks with low β, quality stocks, stocks that pay high dividends (CMA), etc.

iii. How could a die-hard CAPM believer hold on to CAPM despite the evidence above?
ANSWER: Roll's critique: tests "rejecting" CAPM rely on using stocks traded on the stock exchange (usually the US stock market). However, the universe of investment opportunities is much broader than this, and therefore the true market portfolio different from the restricted portfolio based on the stock market alone. Therefore, the restricted portfolio is most likely not on the efficiency frontier. Moreover, the true SML will be different from the estimated SML (which, again, is based on a restricted set of assets). This makes the empirical tests invalid.

2. **Equity premium [45%]**: Consider an individual with preferences $U(c_1, c_2) =$

$u(c_1) + \beta u(c_2)$, where the utility function u is constant relative risk aversion (CRRA);

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma},$$

where $\gamma \geq 0$. The individual has an initial wealth W_0 and has no other income.

(a) [5%] Show that the parameter γ is the relative risk aversion for this individual

ANSWER:

$$RRA = -c \cdot \frac{u''(c)}{u'(c)} = -c \cdot \frac{\gamma^2 c^{-\gamma-1}}{-\gamma c^{-\gamma}} = \gamma$$

(b) [10%] Assume that in period 1 the individual can invest in stocks and bonds, where stocks pay a risky return r_s in period 2 and bonds have a safe return r_f in period 2.

i. Write down the optimization problem for the individual.

ANSWER:

$$\begin{aligned} & \max E\{u(c_1) + \beta u(c_2)\} \\ & \text{subject to} \\ W_0 &= c_1 + S + B \\ c_2 &= (1+r_s)S + (1+r_f)B \end{aligned}$$

ii. Show that the first-order conditions for this individual can be expressed as

$$\begin{aligned} 1 &= E\left\{\frac{\beta u'(c_2)}{u'(c_1)} \cdot (1+r_f)\right\} \\ 1 &= E\left\{\frac{\beta u'(c_2)}{u'(c_1)} \cdot (1+r_s)\right\}. \end{aligned} \qquad (1)$$

ANSWER: Substitute in the budget constraints into the utility function:

$$\max_{S,B} E\{u(W_0 - S - B) + \beta u((1+r_s)S + (1+r_f)B)\}.$$

Take the first-order conditions with respect to S and B,

$$\begin{aligned} 0 &= -u'(W_0 - S - B) + E\{\beta u'((1+r_s)S + (1+r_f)B)(1+r_s)\} \\ 0 &= -u'(W_0 - S - B) + E\{\beta u'((1+r_s)S + (1+r_f)B)(1+r_f)\} \end{aligned}$$

Rearranging yields the desired Euler equations.

(c) [15%] Assume that c_1 and c_2 are aggregate consumption and that all households have the same CRRA utility function U above. Moreover, assume that the aggregate consumption growth, c_{t+1}/c_t, and the risky return r_2 are jointly log-normal. Show that the equity premium can be expressed as

$$E\left(r_s\right) - r_f \approx \gamma \cdot cov\left(\ln\left(\frac{c_{t+1}}{c_t}\right), r_s\right) \tag{2}$$

ANSWER: Assume that the stochastic processes for endowments (and, hence, consumption) and rates of return are exogenous and that consumption growth, c_{t+1}/c_t, and return on stocks, $(1+r_{s,t+1})$, are jointly log normal. Namely, define the variables $\{\epsilon_{c,t+1}, \epsilon_{s,t+1}\}$ such that

$$\frac{c_{t+1}}{c_t} = \bar{\mathbf{c}}_\Delta \exp\left(\epsilon_{c,t+1} - \sigma_c^2/2\right), \tag{3}$$

$$1 + r_{s,t+1} = (1+\bar{r}_s)\exp\left(\epsilon_{s,t+1} - \sigma_s^2/2\right). \tag{4}$$

The variables $\epsilon_{c,t+1}$ and $\epsilon_{s,t+1}$ are jointly normally distributed with zero means and variances $\{\sigma_c^2, \sigma_s^2\}$ and $\bar{c}_\Delta$ and $\bar{r}_s$ represent average consumption growth and expected return on stocks, repsectively. Recall that if $x \sim N(\mu, \sigma^2)$ then $E\{\exp(x)\} = \exp(\mu + \sigma^2/2)$. Substituting (3)-(4) into (??) yields

$$\begin{aligned}
1 &= \beta E\left[(1+r_{s,t+1})\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\right] \\
&= \beta(1+\bar{r}_s)\bar{c}_\Delta^{-\gamma} E\{\exp[\epsilon_{s,t+1} - \sigma^2/2 - \gamma(\epsilon_{c,t+1} - \sigma_c^2/2)]\} \\
&= \beta(1+\bar{r}_s)\bar{c}_\Delta^{-\gamma}\exp[(1+\gamma)\gamma\sigma_c^2/2 - \gamma cov(\epsilon_s, \epsilon_c)].
\end{aligned}$$

Taking logarithms on both sides yields

$$\log(1+\bar{r}_s) = -\log(\beta) + \gamma\log(\bar{c}_\Delta) - (1+\gamma)\gamma\sigma_c^2/2 + \gamma cov(\epsilon_s, \epsilon_c),\ i = s, b. \tag{5}$$

Similarly, the Euler equation for bonds yields

$$\begin{aligned}
1 &= \beta E\left[(1+r_b)\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\right] \\
&= \beta\left(1+r_f\right)\bar{c}_\Delta^{-\gamma}\exp[(1+\gamma)\gamma\sigma_c^2/2] \\
&\Rightarrow \\
\log(1+r_f) &= -\log(\beta) + \gamma\log(\bar{c}_\Delta) - (1+\gamma)\gamma\sigma_c^2/2
\end{aligned}$$

Finally, the equity premium can be derived as follows:

$$\log(1+\bar{r}^s) - \log(1+r_f) = \gamma \cdot cov(\epsilon_s, \epsilon_c).$$

Using $\log(1+r) \approx r$ yields the result.

(d) [7%] Give an economic interpretation of the formula for the equity premium implied by the model, i.e., equation (2).

ANSWER: Stocks are risky and bonds are safe. If the return on stocks, r_s, is positively correlated with aggregate consumption growth, then the return on stocks will be low when the household needs it the most, i.e., when consumption growth is low and the marginal utility of consumption threfore is high. The equity premium is threfore larger the higher is the covariance between consumption growth and risky return. This effect of covariance is proportional to risk aversion γ because the risk aversion determines how averse the individuals are to having low consuption when the return is low. Or, more precisely, the risk aversion determines how much the marginal utility changes in response to a change in consumption.

(e) [5%] Mehra and Prescott (1985) documented the following statistics for the 1889-1980 period: $E(r_s) = 7\%$, $E(r_f) = 1\%$, and the following covariance matrix,

	r_s	r_f	$\ln\left(\frac{c_{t+1}}{c_t}\right)$
r_s	0.0274	0.0010	0.0022
r_f		0.0031	−0.0002
$\ln\left(\frac{c_{t+1}}{c_t}\right)$			0.0013

Explain why these empirical observations are puzzling in light of the equity premium implied by the model, equation (2).

ANSWER: Applying the formula in equation (2) yields the following exprssion for risk aversion γ:

$$\gamma \approx \frac{E(r_s) - r_f}{cov\left(\ln\left(\frac{c_{t+1}}{c_t}\right), r_s\right)} = \frac{0.07 - 0.01}{0.0022} = 27$$

This is an extremely large number compared to the risk aversion implied by experiments where individuals are asked to reveal tradeoffs between risk and returns in lotteries.

(f) [3%] Assume that a new asset is introduced and that this asset is negatively correlated with consumption growth, $\frac{c_{t+1}}{c_t}$. Will the expected return on this asset be lower or higher than the safe interest rate r_f? Give an intuition for your argument.

ANSWER: the return on this asset will be lower than r_f. The reason is that this asset serves as insurance for consumption volatility. Insurance means that the return will be low when consumption growth is high (and, hence, when marginal utility is low) and the return be high when the

need is large, i.e., when consumption growth is low and the marginal utility is high.

3. **Option pricing [20%]**:

   Consider a two-period binominal model for an asset. The price of the asset is $S = 1$ in the first period. Assume that the interest rate is zero ($r = 0$). Moreover, between period 1 and 2 the share price of the asset will either increase with 10% (which happens with probability 0.6) or fall with 10% (which happens with probability 0.4). The asset does not pay any dividends. Consider a call option with a strike price $K = 1$ in the second period.

   (a) [1%] When (i.e., in which states) should the option be exercised?
   ANSWER: Exercise option when $S_1 > K$. Namely, when asset increases in value.

   (b) [9%] Calculate the replicating portfolio. Explain why the replicating portfolio (for a call option) in the binominal model will always involve leveraged positions, i.e., debt ($B \leq 0$).
   ANSWER:

$$\begin{aligned} r &= 0 \\ u &= 1.1 \\ d &= 0.9 \\ p &= \frac{e^r - d}{u - d} = \frac{1 - 0.9}{1.1 - 0.9} = \frac{1}{2} \end{aligned}$$

   This implies

$$\begin{aligned} c_u &= \max\{0, 1.1 - 1\} = 0.1 \\ c_d &= \max\{0, 0.9 - 1\} = 0 \end{aligned}$$

   The replicating portfolio is given by

$$\begin{aligned} u \cdot S\Delta + B &= c_u \\ d \cdot S\Delta + B &= c_d, \end{aligned}$$

   implying

$$\begin{aligned} 1.1 \cdot \Delta + B &= 0.1 \\ 0.9 \cdot \Delta + B &= 0 \end{aligned}$$

   Solve these equations and obtain the replicating portfolio:

$$\begin{aligned} B &= -0.9 \cdot \Delta \\ B &= 0.1 - 1.1 \cdot \Delta \\ &\Rightarrow \\ \Delta &= 0.5 \\ B &= -0.45 \end{aligned}$$

(c) [5%] Calculate the value of the option c in the first period.
ANSWER:

$$c = \frac{1}{e^r} \left(p c_u + (1 - p) \, c_d \right) = \frac{0.1}{2} = 0.05$$

(d) [5%] Why is the probability of a price increase irrelevant for the price of the option?
ANSWER: the statistical probability of the stock price increasing (which is 60% in this example) is irrelevant for the price of the option because the probability of a price increase is already reflected in the price of the stock (which is $S = 1$ in this example). The replicating portfolio is constructed so that its return matches the return on the option, irrespectively of the actual outcome.