1 CAPM

Consider an economy where individuals have linear-quadratic preferences. Individuals are endowed with initial wealth which they invest in a set of risky assets.

1. Assume first that there is no riskfree asset. Explain the set of portfolios that households would choose in equilibrium. You may draw a diagram.

   ANSWER: Households will maximize expected return given a standard deviation of their portfolio. This maximization problem gives rise to the mean-variance efficiency frontier. Note that only the upper half of the mean-variance frontier is chosen in equilibrium. Draw the efficiency frontier,

2. Explain how the set of portfolios households will choose, changes when one of the assets is risk free. You may draw a diagram.

   ANSWER: When a risk free asset exists, the two-fund separation applies: all individuals will choose a combination of the risk free asset and the market portfolio. The efficiency frontier then changes to the capital market line, which is the (linear) set of portfolios starting at the risk free rate and going through the market portfolio. Draw the capital market line.

3. Suppose the riskfree interest rate is $r_f = 0$. Suppose the market portfolio is the set of all stocks in the world and that this portfolio has an expected return of $E(R_M) = 5\%$ and a standard deviation of 10%. Consider a stock in Telenor. Suppose Telenor has a return R_i with standard deviation $\sigma(R_i) = 30\%$ and a covariance with the market portfolio of $cov(R_i, R_M) = 0.02$.

   (a) What is the expected return $E(R_i)$ for Telenor stocks?
   ANSWER: Using CAPM, $E(R_i) = r_f + \beta_i (E(R_M) - r_f) = r_f + \frac{cov(R_M,R_i)}{var(R_M)} (E(R_M) - r_f)$ which implies $E(R_i) = 0 + \frac{0.02}{(0.1)^2}(0.05 - 0) = 10\%$

   (b) Note that the variance of Telenor's return is $\sigma^2(R_i) = 0.09$. What is the specific risk (or idiosyncratic risk) and market risk (or systematic risk) for Telenor?
   ANSWER: The systematic risk is $cov(R_i, R_M) = 0.02$ while the idiosyncratic risk is $\sigma^2(R_i) - cov(R_i, R_M) = 0.09 - 0.02 = 0.07$

4. Suppose you observe that over time, the covariance with the market portfolio remains $cov(R_i, R_M) = 0.02$ but the average return on Telenor is 15% (and larger than the answer you found in question 3). What would you conclude about the CAPM model?

   ANSWER: This observation implies that Telenor lies above the Security Market Line. This violates the central implication of CAPM. The conclusion is that CAPM must be rejected in this case. Otherwise, all investors would flock to Telenor and bid down the price of Telenor until it lies on the SML

2 Portfolio choice

Suppose CAPM holds. There are 3 risky assets in the market and one risk free asset. The variance-covariance matrix of the returns on the risky assets is

Covariance	A	B	C
A	0.0064	0.0036	0.0012
B	0.0036	0.0144	0.0036
C	0.0012	0.0036	0.0225

The mean returns are

	$E(R_i)$
Asset A	0.10
Asset B	0.20
Asset C	0.30

The market portfolio is $0.1A + 0.4B + 0.5C$, and the risk-free rate is 0.03.

1. Show that the market portfolio has mean return of 0.24 and a standard deviation of 0.0992 (you can do the rest of the exercises even if you struggle with this question).

   ANSWER:

$$\begin{aligned} E(r_M) &= 0.1 * E(r_A) + 0.4 * E(r_B) + 0.5 * E(r_C) \\ &= 0.1 * 0.1 + 0.4 * 0.2 + 0.5 * 0.3 = 0.24 \end{aligned}$$

$$\begin{aligned} var(r_M) &= var(0.1 * r_A + 0.4 * r_B + 0.5 * r_C) \\ &= x_A^2 * var(r_A) + x_A^2 * var(r_A) + x_A^2 * var(r_A) \\ &\quad + 2x_A x_B * cov(r_A, r_B) + 2x_A x_C * cov(r_A, r_C) + 2x_C x_B * cov(r_B, r_C) \\ &= (0.0992)^2 \end{aligned}$$

2. If an investor requires a return of 14%, what is the optimal allocation of her wealth among all assets? And what level of risk does she bear?

   ANSWER:

$$\begin{aligned} E(r_p) &= \omega r_f + (1-\omega) E(r_M) \\ &\Rightarrow \\ \omega &= \frac{E(r_M) - E(r_p)}{E(r_M) - r_f} = \frac{0.24 - 0.14}{0.24 - 0.03} = \frac{10}{21} \approx 0.476 \end{aligned}$$

   Allocate a share 10/21 in the risk free asset and 11/21 in the market portfolio. The portfolio risk is

$$\sigma_p = (1-\omega)\sigma_m = \frac{11}{21} * 0.0992 \approx 0.0520$$

3. What if the required return is 36%? (Assume that the investor can borrow and lend at the risk-free rate).

   ANSWER: Similarly,

$$\omega = \frac{E(r_M) - E(r_p)}{E(r_M) - r_f} = \frac{0.24 - 0.36}{0.24 - 0.03} \approx -0.571$$

   and a share 1.571 in stocks; and $\sigma_p = 1.571 * 0.0992 \approx 0.1558$. It means if I have 100 euros, I will borrow 57 euros at risk-free rate, and put 157 euros on the market portfolio.

4. Calculate the Sharpe Ratio of the market portfolio. Consider another portfolio $0.7A + 0.1B + 0.2C$, what is the expected return of this portfolio? Explain why – in equilibrium – the Sharpe ratio of the alternative portfolio must be lower than the Sharpe ratio.

   ANSWER: Sharpe ratio is

$$SR_M = \frac{E(r_M) - r_f}{\sigma(r_M)} = \frac{0.24 - 0.03}{0.992} \approx 0.2117$$

   For another portfolio 0.6A+0.3B+0.1C, redo calculation in question a with the new weights, $E(r_{new}) = 0.15$; $\sigma_{new} = 0.0740$; $SR_{new} = 1.62$, smaller than that of the market portfolio.

   Remark: Without doing any calculation, you should know that the market portfolio has the highest Sharpe ratio.

5. Compute the β for each asset.

   ANSWER: First, calculate the covariance of each asset with the market portfolio:

$$\begin{aligned} cov(r_A, r_M) &= cov(r_A, 0.1 * r_A + 0.4 * r_B + 0.5 * r_C) \\ &= 0.1 * var(r_A) + 0.4 * cov(r_A, r_B) + 0.5 * cov(r_A, r_C) \\ &= 0.00268 \end{aligned}$$

$$\beta_A = \frac{cov(r_A, r_M)}{var(r_M)} = \frac{0.0028}{(0.0992)^2} \approx 0.27$$

   Similarly, $\beta_B = 0.80$; $\beta_C = 1.30$.

6. Suppose there existed an asset with $\beta = 1.5$. Using the security market line, what would be the expected return on this asset?

   ANSWER:

$$\begin{aligned} E(r_i) &= r_f + \beta_i (E(r_M) - r_f) \\ &= 0.03 + 1.5 * (0.24 - 0.03) = 34.5\% \end{aligned}$$

3 Consumption-based asset pricing

Consider a representative agent economy where all households consume the same amount c_t (and this amount c_t can vary over time). Households have preferences

$$\max E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) \right\}.$$

There exists many stocks with stochastic returns and a risk-free asset with return r_f. Define the return on the stock R_{it} as

$$R_{it} = \frac{P_{it} + D_{it}}{P_{i,t-1}},$$

where P_{it} is the price of the asset in period t and D_{it} is the dividend of the asset.

1. Explain in words why households will invest such that in equlibrium the following equation is satisfied:

$$P_{it} = E_t \left\{ \beta \frac{u'(c_{t+1})}{u'(c_t)} \left(P_{i,t+1} + D_{i,t+1}\right) \right\}$$

   ANSWER: 1. When choosing optimal portfolio, the households maximize utility. This implies that their intertermporal first-order condition is satisfied. The equation reflects this first-order condition. It trades off the cost of investing one more unit in an asset - the marginal utility of current consumption - against the expected gain of investing in the asset - expected value of return times marginal utility next period.

2. Suppose one stock (Forest Harvest) has a stream of future dividends which is completely independent of aggregate consumption. Note that in this case the return on Forest Harvest, R_{it}, will be independent of c_t. What will be the expected return for Forest Harvest? Explain your answer.

   ANSWER: Since the return on Forest Harvest is independent of future consumption, the expected return must be the same as the risk-free rate. This can be seen by calculating the expectation of the right-hand side of the equation:

$$\begin{aligned}
P_{it} &= E_t \left\{ \beta \frac{u'(c_{t+1})}{u'(c_t)} \left(P_{i,t+1} + D_{i,t+1}\right) \right\} \\
&= E_t \left\{ \beta \frac{u'(c_{t+1})}{u'(c_t)} \right\} * E_t \left\{ P_{i,t+1} + D_{i,t+1} \right\} \\
&\Rightarrow \\
E_t \left\{ R_{i,t+1} \right\} &= E_t \left\{ \frac{P_{i,t+1} + D_{i,t+1}}{P_{it}} \right\} = \frac{1}{E_t \left\{ \beta \frac{u'(c_{t+1})}{u'(c_t)} \right\}} = 1 + r_f
\end{aligned}$$

The last equation follows from the equation from the risk-free asset:

$$\begin{aligned} 1 &= E_t\left\{\beta\frac{u'(c_{t+1})}{u'(c_t)} * (1+r_f)\right\} \\ &\Rightarrow \\ \frac{1}{1+r_f} &= E_t\left\{\beta\frac{u'(c_{t+1})}{u'(c_t)}\right\} \end{aligned}$$

3. Suppose preferences are CRRA, $u(c) = c^{1-\gamma}/(1-\gamma)$, and that the consumption growth c_{t+1}/c_t and returns on stocks are all jointly log-normal. Moreover, suddenly news arrive that announces that future dividends for Forest Harvest will become highly correlated with aggregate consumption, while the expected value of the dividends will remain unchanged. What will happen to the price of the stock of Forest Harvest and to the expected return on Forest Harvest stocks?

   ANSWER: Since returns are jointly log normal, expected return is risk free rate plus risk aversion times covariance between consumption growth and return. With a higher correlation between consumption growth and return, the expected return must increase. Since the expected future dividends are unchanged, this must imply that the price of the stock falls when the news arrive.

4. Suppose households have CRRA risk aversion $\gamma = 5$. Moreover, their stochastic process for consumption is equal to the process for aggregate consumption in the U.S. and the process return on stocks are the same as for the U.S. economy, including an expected return on stocks of $E(R_{M,t}) = 7\%$. Explain why an equilibrium in this economy can be expected to deliver a return on bonds which is much larger than the average empirical return on bonds in the U.S. (which is close to 1%).

   ANSWER: The expected equity premium (expected return on stocks minus the risk free rate) will be very low because the risk aversion is as low as 5. Since aggregate consumption growth is very smooth and consumption growth is not very highly correlated with returns on stocks, the risk aversion must be extremely high in order for the expected equity premium to be equal to its empirical counterpart.