Final Exam ECON4510 «Finance Theory»

Solution: Answers will follow. For instructional purposes, these may be more elaborate than what was expected from the students writing the final exam.

(10 points) 1. *Expected equity return, average cost of capital, and leverage*

You have a firm that uses only equity to finance its activities. Its expected returns are 10%. Assume the firm instead considers the following financing structures:

- 10% debt, 90% equity;
- 50% debt, 50% equity;
- 90% debt, 10% equity.

For simplicity, assume that the cost of debt is 5% and the same in all three cases

(a) What are the firm's expected returns on its equity for each of the financing structures the firm considers?

(b) What is the (weighted) average cost of capital in each of the three cases cases?

Solution:

(a) For pedagogical purposes and without loss of generality, assume the the investment is 1. The firm's equity investment will then be 0.9, 0.5, and 0.1, respectively, and the debt payments will be $0.1 \cdot 1.05$,$0.5 \cdot 1.05$, and $0.9 \cdot 1.05$, respectively.

The expected return on the equity investment will be

$$
\begin{aligned}
\mathbf{E}[R^e] &= \frac{1.1 - 0.1 \cdot 1.05}{0.9} = 0.1056 = 10.56\% \\
\mathbf{E}[R^e] &= \frac{1.1 - 0.5 \cdot 1.05}{0.5} = 0.15 = 15\% \\
\mathbf{E}[R^e] &= \frac{1.1 - 0.9 \cdot 1.05}{0.1} = 0.55 = 55\%
\end{aligned}
$$

(b) The (weighted) average cost of capital is

$$
\begin{aligned}
&0.9 \cdot 10.56\% + 0.1 \cdot 5\% = 10\% \\
&0.5 \cdot 15\% \quad\;\; + 0.5 \cdot 5\% = 10\% \\
&0.1 \cdot 55\% \quad\;\; + 0.9 \cdot 5\% = 10\%
\end{aligned}
$$

(15 points) 2. *Capital budgeting*

Your organization has two potential projects with the following projected cash flows:

"*Project A*":

	Year 0	Year 1	Year 2	Year 3	Year 4
Initial investment	-1,000,000				
Sales		120,000	700,000	900,000	800,000
Cost of goods sold		100,000	300,000	400,000	300,000

"*Project B*":

	Year 0	Year 1	Year 2	Year 3	Year 4	Year 5
Initial investment	-1,000,000					
Sales		800,000	800,000	800,000	800,000	
Cost of goods sold		400,000	400,000	400,000	400,000	
Salvage costs						-400,000

Salvage costs are the costs involved in salvaging goods or property. Salvage value would have been the amount that an asset is estimated to be worth at the end of its useful life.

The risk-free rate is 2%, and the expected market risk premium is 5%, and The beta of "Project A" is 1.8, and the beta of "Project B" is 2.0.

Your organization has only $ 1,000,000 available for investment in this year's budget, so it has to decide which of the two projects to choose.

(a) What is the cost of capital for each of the potential projects?

(b) Which project should your organization choose?

(c) Which project is chosen based on the payback rule?

Solution:

(a) The cost of capital for "Project A" is

$$2.0\% + 1.8 \cdot 5.0\% = 11.0\%$$

while the cost of capital for "Project B" is

$$2.0\% + 2.0 \cdot 5.0\% = 12.0\%$$

("Project B" has a higher cost of capital since it has higher systematic risk).

(b) Your organization should choose the project with the higher NPV (it can afford either A or B, but not both at the same time).

"Project A" has an NPV of $37,628.17$, while "Project B" has an NPV of $-12,031.00$, so your organization should choose "Project A". ("Project B" should be rejected even in the absence of capital constraints, since it is negative NPV).

(c) Project A's payback period is 3.16 years, while Project B's payback period is 2.5 years. The payback rule tells us B should be preferred to A, which is actually the wrong decision in this case. The payback rule ignores, in general, the time value of money and hence the final cash flows, and, in this particular case, the large negative cash flow at the end of Project B's life.

(10 points) 3. *Time value of money and the yield curve*

(a) You have three risk-free zero-coupon bonds. They all have a face value of NOK 1,000. The first bond is a one-year bond and has a price of NOK 950. The second is a two-year bond and has a current price of NOK 900. The third one is a three-year bond and has a current price of NOK 800. Compute the yield curve up to three years based on this information.

(b) You have a three-year bond with annual coupons paid at the end of the year, the next coupon one year from now. The coupon rate is 6%, and the face value is NOK 1,000. What is the price of the bond?

Solution:

(a) The one-year annual yield is

$$\frac{1000}{950} - 1 = 5.26\%$$

The two-year annual yield is

$$\left(\frac{1000}{900}\right)^{\frac{1}{2}} - 1 = 5.41\%$$

The three-year annual yield is

$$\left(\frac{1000}{800}\right)^{\frac{1}{3}} - 1 = 7.72\%$$

(b) The annual coupon is

$$0.06 \cdot 1{,}000 = 60.$$

The price of the bond is the present value of its future cash flows.

$$P_{\text{bond}} = \frac{60}{1.0526} + \frac{60}{1.0541^2} + \frac{1060}{1.0772^3}$$
$$= 60 \cdot \frac{950}{1000} + 60 \cdot \frac{900}{1000} + 1060 \cdot \frac{800}{1000} = 959$$

(10 points) 4. *Expected returns, and risk premia*

You have the following information about assets A, B, C and D:

		Price next year		
Asset	Current price	Good economy	Average economy	Bad economy
A	100	108	104	98
B	205	208	206	204
C	320	304	306	320
D	115	138	119	100

A, B and D are shares that are not expected to pay any dividends over the next 12 months. C is a share that is expected to pay a dividend of 4 per share if the economy is good or average. (The dividend has the same time horizon as the "price next year".)

The probabilities of having a good/average/bad economy over the next year are generally thought to be 25%, 50% and 25% respectively.

(a) What are the expected returns on each of the assets?

(b) If the annual interest rate is 1%, what are their risk premia?

Solution:

(a) The expected prices for A, B, C and D next year are as follows:

$$0.25 \cdot 108 + 0.5 \cdot 104 + 0.25 \cdot 98 = 103.5$$
$$0.25 \cdot 208 + 0.5 \cdot 206 + 0.25 \cdot 204 = 206$$
$$0.25 \cdot 304 + 0.5 \cdot 306 + 0.25 \cdot 320 = 309$$
$$0.25 \cdot 138 + 0.5 \cdot 119 + 0.25 \cdot 100 = 119$$

The expected returns are:

$$\mathbf{E}[r_A] = \frac{103.5 - 100}{100} = 3.5\%$$
$$\mathbf{E}[r_B] = \frac{206 - 202}{202} = 1.98\%$$
$$\mathbf{E}[r_C] = \frac{309 + 0.75 \cdot 4 - 320}{320} = -2.5\%$$
$$\mathbf{E}[r_D] = \frac{119 - 115}{115} = 3.48\%$$

(b) The risk premia are therefore 2.5%, 0.98%, -3.5% and 2.48%.

(10 points) 5. *Dividends and repurchases*

You own 2% of a firm worth NOK 20,000,000.

(a) Suppose NOK 2,000,000 is paid out as dividend. How much do you get? How much are your shares worth after the distribution of the dividend?

(b) The firm repurchases 10% of its shares (it pays NOK 2,000,000 for them). You decide to sell 10% of your shares back to the firm. What is the market value of your wealth? How about when you sell all your shares – or when you do not sell any of them?

Solution:

(a) If the firm pays dividends, you get

$$2\% \cdot \text{NOK } 2{,}000{,}000 = \text{NOK } 40{,}000$$

Your shares are now worth NOK 360,000 (2% of the remaining "slice" of the firm: 20,000,000 - 2,000,000 = NOK 18,000,000)

- Before the dividend distribution the market value of your wealth was NOK 400,000 (2% of NOK 20,000,000)
- After the distribution you have NOK 40,000 in cash and your shares are worth NOK 360,000. The total market value of your wealth is unchanged at NOK 400,000

(b) If you sell 10% (a "proportional" share of what you hold), you get NOK 40,000 in cash for 10% of your shares and still own shares worth NOK 360,000. Your portfolio is identical to what you had when the firm was paying dividends. If you sell all your shares, then you get NOK 400,000 in cash. If you don't sell your shares, you now own 2/90 of a firm worth NOK 18,000,000. Your shares are thus worth NOK 400,000 – the market value of your wealth is exactly the same as before.

(10 points) 6. *Expected future returns*

The value of a financial asset (or a composite of financial assets, like an index) is the net present risk-adjusted value of all future cash flows

$$P_0 = \text{Div}_0 + \frac{\text{Div}_1}{1+r_E} + \frac{\text{Div}_2}{(1+r_E)^2} + \frac{\text{Div}_3}{(1+r_E)^3} + \frac{\text{Div}_4}{(1+r_E)^4} + \cdots = \sum_{n=0}^{\infty} \frac{\text{Div}_n}{(1+r_E)^n}$$

If we assume that dividends grow at a constant (and known) rate μ such that

$$\text{Div}_1 = \text{Div}_0(1+\mu),\ \text{Div}_2 = \text{Div}_0(1+\mu)^2,\ \text{and so on,}$$

show that

$$r_E = \frac{\text{Div}_0}{P_0} + \mu$$

Remember that the sum of an infinite geometric series is

$$s = a + ag + ag^2 + ag^3 + ag^4 + \cdots = \sum_{k=0}^{\infty} ag^k = \frac{a}{1-g},\ \text{for } |g| < 1.$$

and for x_1 and x_2 small

$$(1+x_1)\cdot(1+x_2) \approx 1 + x_1 + x_2$$

Solution:

$$\begin{aligned} P_0 &= \text{Div}_0 + \frac{\text{Div}_0\cdot(1+\mu)}{1+r_E} + \frac{\text{Div}_0\cdot(1+\mu)^2}{(1+r_E)^2} + \frac{\text{Div}_0\cdot(1+\mu)^3}{(1+r_E)^3} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{\text{Div}_0\cdot(1+\mu)^n}{(1+r_E)^n} \\ &= \frac{\text{Div}_0}{r_E - \mu} \end{aligned}$$

Reorganizing to get

$$r_E = \frac{\text{Div}_0}{P_0} + \mu$$

(15 points) 7. *Returns and risk premiums.* Consider the asset prices and payoffs

$$\begin{aligned} \text{Asset 1}: &\quad p_1 = 3/4, \ \tilde{x}_1(1) = 1, \ \tilde{x}_1(2) = 1 \\ \text{Asset 2}: &\quad p_2 = 1, \quad \tilde{x}_2(1) = 1, \ \tilde{x}_2(2) = 2. \end{aligned}$$

(a) What are the state prices? Is $(p, \tilde{x})$ arbitrage free?

(b) What are the returns on the two assets?

(c) If $\text{prob}_1 = \text{prob}_2 = 1/2$, what are the expected returns?

(d) What is the stochastic discount factor? Why does the second asset have a higher excess return?

(e) What are the risk-neutral probabilities? Why does the second asset have a higher excess return?

Solution:

(a) The state prices are

$$q_1 = \frac{1}{2} \quad \text{and} \quad q_2 = \frac{1}{4}$$

Since both are positive, the system is arbitrage free.

(b)

Ex post return of asset 1 if state 1 is realized :

$$R = \frac{1}{\frac{3}{4}} = \frac{4}{3}$$

Ex post return of asset 1 if state 2 is realized :

$$R = \frac{1}{\frac{3}{4}} = \frac{4}{3}$$

Ex post return of asset 2 if state 1 is realized :

$$R = \frac{1}{\frac{1}{1}} = 1$$

Ex post return of asset 2 if state 2 is realized :

$$R = \frac{2}{1} = 2$$

(c) *Expected* returns are 4/3 and 3/2, respectively.

Asset 2 has an expected risk premium of $3/2 - 4/3 = 1/6$.

(d) stochastic discount factors

$$m_1 = \frac{\frac{1}{2}}{\frac{1}{2}} = 1 \quad \text{and} \quad m_2 = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

Asset 2 has higher expected return because it pays off more in state 2, where m is lower.

(e) Risk-neutral probabilities are connected to the stochastic discount factor by

$$\text{prob}_i^* = \text{prob}_i \cdot m(\omega_i) \cdot R_f$$

The risk-free rate is the inverse of the conditional expectation of the stochastic discount factor

$$R_f = [\mathrm{E}(m)]^{-1} = \frac{1}{\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2}} = \frac{4}{3}$$

so

$$\text{prob}_1^* = \frac{1}{2} \cdot 1 \cdot \frac{4}{3} = \frac{2}{3}$$
$$\text{prob}_2^* = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{4}{3} = \frac{1}{3}$$

Asset 2 has higher expected return because it pays off more in state 2, where prob^* is lower.

(20 points) 8. *The risk-free rate and intertemporal consumption choice*

Consider the two-period decision problem: choose consumptions (c_0, c_1) to maximize utility

$$U(c_0, c_1) = \frac{c_0^{1-\rho}}{1-\rho} + \beta \frac{c_1^{1-\rho}}{1-\rho}$$

subject to the budget constraint $c_0 + \frac{1}{R_f} c_1 \leq y_0$ (that is, there's no date-1 income).

(a) What is the Lagrangian function associated with this problem? What are the first-order conditions?

(b) Express consumptions as functions of income y_0 and the risk-free rate R_f.

(c) How does date-0 consumption c_0 depend on R_f?

(d) How does saving depend on R_f? On the interest rate?

(e) *For given* c_0 and c_1, show that the solution to the agent's optimization problem implies that R_f must be lower when the elasticity of intertemporal substitution is higher. Relate to the so-called "risk-free rate puzzle".

Solution:

(a)

$$\mathscr{L} = \frac{c_0^{1-\rho}}{1-\rho} + \delta \frac{c_1^{1-\rho}}{1-\rho} - \lambda \left(c_0 + \frac{1}{R_f} c_1 - y_0 \right)$$

FOCs

$$\begin{aligned} c_0 &: \; c_0^{-\rho} - \lambda = 0 \\ c_1 &: \; \delta \cdot c_1^{-\rho} - \lambda \frac{1}{R_f} = 0 \\ \lambda &: \; c_0 + \frac{1}{R_f} c_1 - y_0 = 0 \end{aligned}$$

so

$$\delta \cdot R_f \cdot \left(\frac{c_1}{c_0} \right)^{-\rho} = 1$$

(b)

$$c_0 = \left(\frac{1}{\delta R_f} \right)^{\frac{1}{\rho}} c_1$$

so

$$c_0 = \left(\frac{1}{\delta R_f} \right)^{\frac{1}{\rho}} (y_0 - c_0) R_f$$

and

$$c_0 = \frac{y_0}{1 + \delta^{1/\rho} \cdot R_f^{1/\rho - 1}}$$

(c) Three cases

i. If $\rho > 1$ then $\partial c_0 / \partial R_f > 0$: The income effect is larger than the substitution effect

ii. If $\rho < 1$ then $\partial c_0 / \partial R_f < 0$: The substitution effect is larger than the income effect

iii. If $\rho = 1$ (log utility) then $\partial c_0 / \partial R_f = 0$: The income effect and the substitution effect cancels each other out

(d) Three cases

i. If $\rho > 1$ then $\partial(y_0 - c_0)/\partial R_f < 0$: The income effect is larger than the substitution effect so the agent can increase c_0 by decreasing savings

ii. If $\rho < 1$ then $\partial(y_0 - c_0)/\partial R_f > 0$: The substitution effect is larger than the income effect

iii. If $\rho = 1$ (log utility) then $\partial(y_0 - c_0)/\partial R_f = 0$: The income effect and the substitution effect cancels each other out

(e)

$$\frac{\partial \frac{1}{\rho}}{\partial R_f} = -\frac{\log \frac{1}{\rho} + \log \frac{c_1}{c_0}}{R_f \cdot \log^2(R_f)} < 0$$

so if R_f increases the elasticity of intermporal substitution must decrease to keep the ratio of c_1 and c_0 unchanged. This is in essence the same as the mechanisms as behind the so-called "risk-free rate puzzle"