Final Exam ECON4510 «Finance Theory»

Solution: Answers will follow. For instructional purposes, these may be more elaborate than what was expected from the students writing the final exam.

(10 points) 1. *Portfolio Choice*

Your aunt asks for investment advice. Currently, she has NOK500, 000 invested in portfolio p, which consists of three stocks. Portfolio p has an expected return of 3.0% and a standard deviation of 12%. Suppose the risk-free rate is 1%, and the tangent portfolio has an expected return of 5% and a standard deviation of 15%.

(a) To maximize his expected return without increasing her volatility, which portfolio would you recommend?

(b) If your aunt prefers to keep her expected return the same but minimize her risk, which portfolio would you recommend?

Solution:

(a) To maintain the volatility at 12%, your aunt would invest $12/15 = 0.8$ of her portfolio in the market/tangent portfolio and $3/15 = 0.2$ in the risk free asset

$$\mathbf{E}[R] = 0.2 \cdot 1\% + 0.8 \cdot 4\% = 3.4\%$$

(b) Alternatively, to keep the expected return equal to 3.0%, her portfolio shares must satisfy

$$\mathbf{E}[R] = (1 - x) \cdot 1\% + x \cdot 4\% = 3.0\%$$

hence $x = 2/3$.

The standard deviation of her portfolio is then $2/3 \cdot 15\% = 10\%$.

(15 points) 2. *Bonds and bond pricing*

You have the following information about several risk-free zero-coupon bonds:

Bond	Years to maturity	Price
A	1	972
B	2	946
C	3	914
D	4	823

All bonds have a face value of NOK 1000.

(a) What will happen to the prices of the zero-coupon bonds as they approach their maturities if market yields remain un-changed?

(b) Derive the term structure of risk-free interest rates based on this information.

(c) If market yields are unchanged – and are as expected based on the yield curve, what will the price of bond C be one year from now?

(d) You have a risk-free bond – bond E – with 3 years to maturity. The bond has a face value of NOK 10,000 and a coupon rate of 7%. The next coupon will be paid one year from now, and the bond pays annual coupons. What is the price of the bond?

Solution:

(a) If market yields remain unchanged, the price of zero-coupon bonds will gradually increase towards the face value until maturity.

(b) The annualized yields are as follows:

Bond	Years to maturity	Price	Yield to maturity
A	1	972	2.88%
B	2	946	2.81%
C	3	914	3.04%
D	4	823	4.99%

(c) If market yields are unchanged, the price of bond C one year from now will be

$$914 \cdot 1.0288 = 940.33$$

(d) The price of bond E is

$$\frac{700}{1.0288} + \frac{700}{1.0281^2} + \frac{10700}{1.0304^3} = 11,123$$

(15 points) 3. *Two stocks*

We are following two listed companies, Value Inc. and Tech Inch. The risk-free rate is 1%, the expected return on the market portfolio in excess of the risk-free rate is 4%, the beta of Value Inc. is 0.75, and the beta of Tech Inc. is 1.5.

(a) What is the equity cost of capital for each company?

(b) Without loss of generality, assume we only consider the next three years. The expected dividends per share for the two companies are

	Year 1	Year 2	Year 3
Value Inc.	100	100	100
Tech Inc.	0	0	340

What is the share price for each company?

(c) What happens to the share prices if the risk-free rate suddenly and unexpectedly increases from 1% to 2%?

(d) Assume now that, in addition, the price of risk increases and the expected return on the market portfolio in excess of the risk-free rates increases to 6%

i. What are now the share prices?

ii. How much have the share prices changed, respectively?

iii. What are now the expected rate of return on each stock, respectively?

Solution:

(a) Equity cost of capital for Value Inc

$$r^e_{\text{Value}} = 1\% + 0.75 \cdot 4\% = 4\%$$

and for Tech Inc

$$r^e_{\text{Tech}} = 1\% + 1.5 \cdot 4\% = 7\%$$

(b) Share price for Value Inc

$$p_{\text{Value}} = \frac{100}{1.04} + \frac{100}{1.04^2} + \frac{100}{1.04^3} = 277.5$$

and for Tech Inc

$$p_{\text{Tech}} = 0 + 0 + \frac{300}{1.07^3} = 277.5$$

(c) After risk-free rate increase, share price for Value Inc

$$p_{\text{Value}} = \frac{100}{1.05} + \frac{100}{1.05^2} + \frac{100}{1.05^3} = 272.3$$

and for Tech Inc

$$p_{\text{Tech}} = 0 + 0 + \frac{300}{1.08^3} = 269.9$$

(d) After price-of-risk increase

i. Share price for Value Inc

$$p_{\text{Value}} = \frac{100}{1.065} + \frac{100}{1.065^2} + \frac{100}{1.065^3} = 264.9$$

and for Tech Inc

$$p_{\text{Tech}} = 0 + 0 + \frac{300}{1.11^3} = 248.6$$

ii. Share-price decline for Value Inc

$$\frac{264.9 - 277.5}{277.5} = -4.5\%$$

and for Tech Inc

$$\frac{248.6 - 277.5}{277.5} = -10.4\%$$

iii. The expected rates of return are 6.5% and 11% respectively.

(15 points) 4. *State prices and related objects.* Consider an economy with three states. State prices and probabilities are

State ω	State Price $q(\omega)$	Probability prob_i	Payoff $\tilde{x}(\omega)$
1	1/3	1/2	1
2	1/3	1/4	2
3	1/3	1/4	3

(a) What is the (stochastic) discount factor in each state?

(b) What is the price of a one-period bond? What is its return?

(c) What are the risk-neutral probabilities? Why are they different from the true probabilities?

(d) Suppose equity is a claim to the dividend in the last column. What is its price? What is the return on equity in each state?

(e) What is the expected return on equity? The risk premium?

Solution:

(a)

State 1 :

$$m(\omega_1) = \frac{q(\omega_1)}{\text{prob}_1} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

State 2 :

$$m(\omega_2) = \frac{q(\omega_2)}{\text{prob}_2} = \frac{\frac{1}{3}}{\frac{1}{4}} = \frac{4}{3}$$

State 3 :

$$m(\omega_3) = \frac{q(\omega_3)}{\text{prob}_3} = \frac{\frac{1}{3}}{\frac{1}{4}} = \frac{4}{3}$$

(b)

$$p^b = \text{E}(m) = \sum_i (\text{prob}_i \cdot m(\omega_i)) = \frac{1}{2} \cdot \frac{2}{3} + \frac{1}{4} \cdot \frac{4}{3} + \frac{1}{4} \cdot \frac{4}{3} = 1$$

and $R_f = 1/p^b$ so

$$R_f = \frac{1}{p^b} = 1$$

(c) Risk-neutral probabilities are connected to the stochastic discount factor by

$$\text{prob}_i^* = \text{prob}_i \cdot m(\omega_i) \cdot R_f$$

so

$$\text{prob}_1^* = \frac{1}{2} \cdot \frac{2}{3} \cdot 1 = \frac{1}{3}$$
$$\text{prob}_2^* = \frac{1}{4} \cdot \frac{4}{3} \cdot 1 = \frac{1}{3}$$
$$\text{prob}_3^* = \frac{1}{4} \cdot \frac{4}{3} \cdot 1 = \frac{1}{3}$$

(d)

$$p = \text{E}(m\tilde{x}) = \frac{1}{2} \cdot \frac{2}{3} \cdot 1 + \frac{1}{4} \cdot \frac{4}{3} \cdot 2 + \frac{1}{4} \cdot \frac{4}{3} \cdot 3 = 2$$

Realized returns, conditional on

realization of state 1 :

$$R = \frac{1}{2} = 0.5$$

realization of state 2 :

$$R = \frac{2}{2} = 1$$

realization of state 3 :

$$R = \frac{3}{2} = 1.5$$

(e) Expected returns

$$\mathbf{E}(R) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{4} \cdot 1 + \frac{1}{4} \cdot \frac{3}{2} = \frac{7}{8} = 0.875$$

Expected excess returns

$$\mathbf{E}(R - R^f) = \frac{7}{8} - 1 = -\frac{1}{8} = -0.125$$

(10 points) 5. *Time-preferences, consumption, and interest rates*

Consider the following deterministic two-period optimization problem

$$\max U(c_0, c_1) = \max\{u(c_0) + \beta u(c_1)\},$$

where β is the time-discount factor subject to the budget constraint

$$c_0 + (1/R)c_1 \leq y_0 + (1/R)y_1.$$

where y_0 and y_1 are outside income in period 1 and period 2, respectively.

If we define saving as $s = y_0 - c_0$, this becomes

$$c_1 \leq Rs + y_1.$$

(a) Set up the Lagrange problem, derive the first-order conditions, and show that

$$\beta \frac{u'(c_1)}{u'(c_0)} R = 1.$$

(b) Assume that

$$u(c) = \frac{c^{1-\alpha}}{1-\alpha}$$

Define (gross) consumption growth

$$g_c \equiv \frac{c_1}{c_0}$$

Show that the real interest rate can be expressed as a function of consumption growth and preference parameters α and β

$$R = \frac{1}{\beta} g_c^{\alpha}.$$

(c) Assume long-run average annual consumption growth rate is 1.8% per year, and we infer from other studies that $\beta = 0.995$, and $\alpha = 1$, what is the real interest rate?
Does this match real yield on long-term bonds? What if $\alpha = 2$? (Be *brief*).

Solution:

(a)

$$\mathcal{L} = u(c_0) + \beta u(c_1) - \lambda\left(c_0 + (1/R)c_1 - y_0 - (1/R)y_1\right)$$

The first-order conditions are

$$u'(c_0) = \lambda$$
$$\beta u'(c_1) = \lambda \frac{1}{R},$$

combining we get the intertemporal optimality condition

$$\beta \frac{u'(c_1)}{u'(c_0)} R = 1.$$

(b) If

$$u(c) = \frac{c^{1-\alpha}}{1-\alpha}$$

then

$$u'(c) = c^{-\alpha}$$

So the intertemporal optimality condition is

$$\beta \left(\frac{c_1}{c_0}\right)^{-\alpha} R = 1.$$

Defining

$$g_c \equiv \frac{c_1}{c_0}$$

and reorganizing we get

$$R = \frac{1}{\beta} g_c^{\alpha}.$$

(c) If $\alpha = 1$,

$$R = \frac{1}{0.995} \cdot 1.018 = 1.023$$

and if $\alpha = 2$,

$$R = \frac{1}{0.995} \cdot 1.018^2 = 1.042$$

Ten-year real yields are hardly positive so the real rate implied by $\alpha = 1$ would be closer than for $\alpha = 2$.

(15 points) 6. *Power utility, certainty equivalents, and risk premia*

Power utility has the form

$$u(c) = \frac{c^{1-\alpha}}{1-\alpha}$$

(a) The certainty equivalent, μ, is the solution to

$$\sum_z p(z)u[c(z)] = \sum_z p(z)u(\mu).$$

where $p(z)$ is the probability distribution over states z.

Show that with power utility the certainty equivalent is

$$\mu = \left(\sum_z p(z)c(z)^{1-\alpha}\right)^{1/(1-\alpha)} = \left[\mathrm{E}\left(c^{1-\alpha}\right)\right]^{1/(1-\alpha)}.$$

(b) The risk premium, Π, is defined as the amount that makes an investor indifferent between a risky lottery and a certain amount

$$\Pi = \mathrm{E}c - \mu$$

Assume this period's consumption is 1 and next period's risky consumption is

$$c = \begin{cases} 1.051 & \text{with probability } 1/2 \\ 0.985 & \text{with probability } 1/2 \end{cases}$$

Show that with these numbers we match approximately the following two moments from data: a mean consumption growth rate of 1.8 percent and standard deviation of 3.3 percentage points.

If her/his risk aversion (α) is 2, what is the certainty equivalent that would make her/him indifferent with the risky lottery?

(c) What is the risk premium? *Briefly* comment on the magnitude and how this relate to the risk premium puzzles, in particular the equity premium puzzle

Solution:

(a) Since

$$\sum_z p(z)u[c(z)] = \sum_z p(z)u(\mu) = u(\mu)$$

then

$$\frac{\mu^{1-\alpha}}{1-\alpha} = \sum_z p(z)\frac{c(z)^{1-\alpha}}{1-\alpha}$$

and

$$\mu^{1-\alpha} = \sum_z p(z)c(z)^{1-\alpha}$$

so

$$\mu = \left(\sum_z p(z) c(z)^{1-\alpha} \right)^{\frac{1}{1-\alpha}}$$

(b) $\mu = 1.0169$

(c) $\Pi = 1.0180 - 1.0169 = 0.0011 = 0.11\%$. This is small compared to observed risk premia in financial markets. This is another example that power utility gives low prices for risk.

(15 points) 7. *Hansen-Jagannathan bound*

(a) Starting from the no-arbitrage condition for all assets i

$$\mathbf{E}\left[mR^i\right] = 1$$

show that

$$\frac{\sigma(m)}{\mathbf{E}[m]} \geq \frac{\mathbf{E}[R^i] - R^f}{\sigma(R^e)}$$

(b) If the average investor has time-separable utility with time-preference parameter β and instantaneous utility function

$$u(c) = \frac{c^{1-\alpha}}{1-\alpha}$$

then

$$m_{t+1} = \beta \left(\frac{c_{t+1}}{c_t} \right)^{-\alpha}$$

Assuming $\mathbf{E}[m_{t+1}] \approx 1$, and with the following historical measures, $\mathbf{E}(R^{\text{market}}) - R^f = 5\%$, and $\sigma(R^{\text{market}}) = 15\%$, what is the lower bound of $\sigma(m)$?

If the gross growth rate of consumption is either 1.051 or 0.985, each with probability 0.5, and $\alpha = 2$, what is $\sigma(m)$?

Comment *briefly*.

(c) What is considered the key insight from the Hansen-Jagannathan bound in terms of diagnostics of the challenge quantitatively accounting for the risk premia in financial markets?

Solution:

(a) From

$$\mathbf{E}\left[mR^i\right] = 1$$

then

$$\mathbf{E}\left[m(R^i - R^f)\right] = 0$$

Defining $R^e \equiv R^i - R^f$

$$\mathbf{E}\left[m(R^e)\right] = 0$$

Taking

$$\mathbf{E}\left[m(R^e)\right] = 0$$

then

$$\mathbf{E}\left[m(R^e)\right] = \mathbf{E}m\mathbf{E}R^e + \rho\sigma(m)\sigma(R^e) = 0$$

and since $-1 \leq \rho \leq 1$

$$\frac{\sigma(m)}{\mathbf{E}[m]} \geq \frac{\mathbf{E}[R^i] - R^f}{\sigma(R^e)}$$

(b)

$$\sigma(m) \geq \frac{\mathbf{E}[R^i] - R^f}{\sigma(R^e)} \cdot \mathbf{E}[m] = \frac{1}{3}$$

$$\sigma(m) = \sqrt{\frac{1}{2}\left(\left(\frac{1}{1.051}\right)^{-2} - \left(\frac{1}{1.018}\right)^{-2}\right) + \frac{1}{2}\left(\left(\frac{1}{0.985}\right)^{-2} - \left(\frac{1}{1.018}\right)^{-2}\right)}$$
$$= 0.06720$$

The pricing kernel is not sufficiently volatile. It does not satisfy the HJ bounds and can not account for the risk premia we observe in financial markets

(c) The Hansen-Jagannathan bound identified the pricing kernel, and, in particular, the volatility of the pricing kernel, to be crucial to account for observed risk premia in financial markets

(5 points) 8. *Campbell-Shiller*

We have the Campbell-Shiller decomposition

$$\underbrace{r_t - \mathbf{E}_{t-1} r_t}_{\text{Returns relative to expectation}} = \underbrace{(\mathbf{E}_t - \mathbf{E}_{t-1})}_{\text{"News" on}} \Bigg[\underbrace{\sum_{j=0}^{\infty} \rho^j \Delta d_{t+j}}_{\text{future dividend growth}} - \underbrace{\sum_{j=1}^{\infty} \rho^j \Delta r_{t+j}}_{\text{discount rates}} \Bigg]$$

Assume that we during the last year experienced higher than expected real returns...

(a) ... if it solely was due to new information about future dividend growth, how has expected returns going forward changed?

(b) ... if it solely was due to new information about discount rates, how has expected returns going forward changed?

Solution:

(a) If it was solely due to new information about future dividend growth, expected returns are *unchanged*

(b) If it was solely due to new information about discount rates, how has expected returns are *lower* than before