YOU MAY ANSWER IN ENGLISH OR NORWEGIAN.

Some advice: Start by reading through the whole exam, and make sure that you allocate time to answering questions you find easy. You can get a good grade even if there are parts of problems that you do not have time to solve. It is better to try to do something on each question than to get bogged down with one question. If you find you are spending too much time on one question, stop working on it and plan to get back to it if you have time at the end. Make sure you state any assumptions you make.

1. Consider an economy where there are two assets, A and B, with expected returns $E\left(\tilde{R}_A\right)$ and $E\left(\tilde{R}_B\right)$, standard deviation of returns σ_A and σ_B, and covariance s_{AB}.

   (a) Use a mean/standard-deviation diagram to illustrate the mean and standard deviation of various portfolios one can obtain by creating portfolios combining these two assets, assuming no restrictions on short selling.

   - **The mean and standard deviation of the portfolio forms a hyperbola that originates in each of the underlying assets.**

   (b) How would the set of combinations of mean and standard deviations for possible portfolios change if the investors could not short sell the assets?

   - **The hyperbola between the two assets would remain unchanged. But due to no short selling the hyperbola does not extend beyond the mean-standard deviation location of each asset.**

   (c) Let p_A and p_B be the prices of shares in assets A and B, respectively, and let x_A and x_B be the total number of shares of the respective assets. Define the "market portfolio" as the portfolio with weights $\omega = p_A \cdot x_A / (p_A \cdot x_A + p_B \cdot x_B)$ on asset A and $1 - \omega$ on asset B. Suppose first that all investors are identical (and do not necessarily have mean-variance preferences) and assume that the asset market is in equilibrium. Explain why asset prices p_A and p_B must be such that it is optimal for all investors to hold the market portfolio.

   - **The central equilibrium condition for a competitive equilibrium is that individuals optimize and markets clear, i.e., that supply equals demand. For some candidate prices p_A and p_B, the supply of each asset is $p_A \cdot x_A$ and $p_B \cdot x_B$, respectively. Suppose it is optimal for households to hold the market portfolio ω and $1 - \omega$. This would**

satisfy the equilibrium conditions. Suppose it is not optimal to hold the market portfolio for some individuals. Since by assumption all individuals are identical, then it must be the case that it is suboptimal for all households to hold the market portfolio. In this case supply would not equal demand so this cannot be an equilibrium.

(d) Assume now that investors have linear-quadratic preferences although investors can differ in risk aversion and wealth. Moreover, assume that investors can purchase a risk-free asset with return r_f. Show that all investors hold exactly the same portfolio of risky assets, although they may differ in their holdings of the risk-free asset.

- **Two-fund separation: With access to a risk-free asset, all mean-standard deviation combinations along the linear line between $(r_f, 0)$ and $\left(E\left(\tilde{R}_M\right), stddev\left(\tilde{R}_M\right)\right)$ become achievable. All these mean-standard deviation combinations (with the exception of the market portfolio) lie to the North-West of the efficiency frontier with two risky assets (derived in point b above). With linear-quadratic preferences households strictly prefer portfolios that lie toward the North-West in the mean-standard deviation diagram (higher mean and lower variance). This implies that the efficiency frontier must be the linear line between the risk free asset and the market portfolio. Thus, any efficient portfolio is comprised of a combination of two portfolios: the risk free rate and the market portfolio, regardless of where on the linear efficiency frontier they might prefer.**

(e) Define $\tilde{R}_M$ as the return on the market portfolio. Write down an expression for the expected return on each asset i as a function of r_f, $E\left(\tilde{R}_M\right)$, $cov\left(R_i, \tilde{R}\right)$, and $var\left(\tilde{R}_M\right)$. Note: you get more points if you *derive* this expression (i.e., motivate how it is obtained).

- **Consider an equilibrium, everyone holds combination of risk free asset and market portfolio. Derive a relation between μ_j, σ_j (of any asset, numbered j) and the economy-wide variables r_f, μ_M, σ_M. As a thought experiment, make a portfolio with a fraction a in asset j and a fraction $1-a$ in the market portfolio. For this portfolio p we have:**

$$\mu_p = a\mu_j + (1-a)\mu_M, \quad \frac{\partial \mu_p}{\partial a} = \mu_j - \mu_M,$$

$$\sigma_p = \sqrt{a^2\sigma_j^2 + (1-a)^2\sigma_M^2 + 2a(1-a)\sigma_{jM}},$$

$$\frac{\partial \sigma_p}{\partial a} = \frac{a\sigma_j^2 - (1-a)\sigma_M^2 + (1-2a)\sigma_{jM}}{\sqrt{a^2\sigma_j^2 + (1-a)^2\sigma_M^2 + 2a(1-a)\sigma_{jM}}},$$

$$\left.\frac{\partial \sigma_p}{\partial a}\right|_{a=0} = \frac{\sigma_{jM} - \sigma_M^2}{\sigma_M}.$$

Use the formula:

$$\frac{d\mu}{d\sigma}\frac{\partial \sigma}{\partial a} = \frac{\partial \mu}{\partial a} \iff \frac{d\mu}{d\sigma} = \frac{\frac{\partial \mu}{\partial a}}{\frac{\partial \sigma}{\partial a}}.$$

Use partial derivatives just found, evaluate at $a = 0$:

$$\left.\frac{\partial \sigma}{\partial a}\right|_{a=0} = \frac{\sigma_{jM} - \sigma_M^2}{\sigma_M}.$$

Plug in and find:

$$\left.\frac{d\mu}{d\sigma}\right|_{a=0} = \frac{\mu_j - \mu_M}{(\sigma_{jM} - \sigma_M^2)/\sigma_M}.$$

This slope of the small hyperbola must equal the slope of the so-called capital market line:

$$\frac{\mu_j - \mu_M}{(\sigma_{jM} - \sigma_M^2)/\sigma_M} = \frac{\mu_M - r_f}{\sigma_M} \iff \mu_j = r_f + (\mu_M - r_f)\frac{\sigma_{jM}}{\sigma_M^2}.$$

Define $\beta_j \equiv \frac{\sigma_{jM}}{\sigma_M^2}$. Then rewrite the expression as

$$E(\tilde{R}_j) - r_f = \beta_j(E(\tilde{R}_M) - r_f),$$

In words, "the expected excess rate of return on asset j equals its beta times the expected excess rate of return on the market portfolio."

2. Consider an economy with many risky assets but no risk free asset. Assume that investors have linear-quadratic preferences.

   (a) Show that a version of CAPM can be derived even without a risk free asset.

      - **Suppose households have mean-variance preferences but there is no riskfree asset. Let p be an efficient portfolio (i.e., p is on the efficiency frontier). Let $ZC(p)$ denote the portfolio with the minimum variance among those portfolios which has zero covariance with p. Then for any portfolio q it can be shown that the following equation holds,**

$$E(\tilde{r}_q) = E\left(\tilde{r}_{ZC(p)}\right) + \beta_{pq}\left[E(\tilde{r}_p) - E(\tilde{r}_{ZC(p)})\right],$$

where β_{pq} is defined as $\beta_{pq} = cov(\tilde{r}_q, \tilde{r}_p)/var(\tilde{r}_p)$. This is very similar to the CAPM equation. But instead of the market portfolio, M, we now have p, which can be any frontier portfolio. And instead of the risk free interest rate, we now have the expected rate of return on the frontier portfolio which has zero covariance with p.

(b) Suppose investors differ in their risk aversion and/or their invested wealth. Explain why investors in this case (i.e., when there is no risk-fee asset) will in general not hold the market portfolio.

- **The efficiency frontier is now the upper part of the hyperbola, starting from the minimum-variance portfolio. The optimal portfolio for a households will be located where the preference indifference curve of the household is tangent with the efficiency frontier. Households with different risk aversion or different wealth to be invested will then choose different optimal portfolios. These portfolios will in general differ from the market portfolio**

3. Suppose you want to test empirically if CAPM holds true. Your plan is to test if assets lie on the Security Market Line.

(a) Derive an empirical strategy to perform this test (explain in words). In particular, explain why it is it common to do such tests for portfolios of assets rather than for individual assets.

- **The central implication of CAPM is that the only aspect of an asset which is relevant for asset pricing is the market beta $\beta_j = cov\left(\tilde{R}_j, \tilde{R}_M\right)/var\left(\tilde{R}_M\right)$ of the asset. Moreover, all assets are located on the (linear) the security market line SML. This implies that the expected return is linear in β_j and that no other aspects of the assets contribute to explaining expected return once β_j has been controlled for. If there is a risk free asset then this implies $E(\tilde{R}_j)-r_f = \beta_j(E(\tilde{R}_M)-r_f)$, and if no risk free asset then r_f is replaced by the expected return on the zero-covariance portfolio. One way to test this implication is to evaluate if the realized return is indeed located on the SML. It is useful to use a portfolio of assets since this the portfolio allows the econometrician to average out the idiosyncratic risk of the individual assets. The standard approach is to use a pre-period to estimate β_j for individual stocks, then sort stocks according to β_j and form portfolios of for example deciles of the β_j of the stocks (or in addition some other aspect such as size). One would then use the following period to estimate the β for each of these portfolios and, finally, some**

future period (e.g. the following 48 months) to evaluate the expected return. The predictions are that (1) this realized return on average is linear in β, and that (2) no other aspect of the portfolios (such as size, price/book value, etc.) contribute to explaining the realized return once β_j has been controlled for.

(b) Suppose you find that your estimate for the expected return on portfolios of assets (where these portfolios are constructed so that assets in each portfolio have roughly the same market β) are roughly linear as a function of the expected β of each portfolio. However, according to this linear function the return on a portfolio with $\beta = 0$ is higher than the risk-free rate. Would this observation constitute a rejection of CAPM? Motivate your answer.

- **This would not necessarily be a rejection of CAPM. While a linear SML which crosses the y-axis above r_f is consistent with Black's zero-beta version of CAPM (although it would be inconsistent with a strict interpretation of the Sharpe-Lintner version of CAPM). For example, it could be that the riskfree rate is not available to investors (due to short-selling constraints, borrowing-lending premia, etc.), in which case the zero-beta CAPM is the relevant model.**

(c) If you wanted to show that CAPM is rejected in Norwegian stock market data, how would you do so in practice (i.e., what would the test look like and what portfolios would you construct)?

- **Follow the approach in 3a above. Estimate β_j for stocks and also collect data on size, price/book, and perhaps recent momentum. For size, construct portfolios with high/low β and small/large. Construct portfolios with zero price, for example long in small stocks and short in large stocks (Small Minus Big). Estimate the realized SML by running a factor regression, and evaluate if size contributes to return over and above β. Same procedure for price/book, etc.**

(d) Would Richard Roll be convinced by your test? Why or why not.

- **Roll's critique is that the prediction of CAPM requires knowledge of the true market portfolio. This includes all assets, also assets not traded on the stock exchange (human capital, real estate, foreign stocks, etc.) "Testing" CAPM using stock market data misrepresents the market return, so a rejection of CAPM is a rejection of the joint hypothesis that CAPM is true and that one has a measurement of the true market return (and that the estimates of the β_j of a stock based on past observations**

is a true representation of the β of the stock also going forward).

4. Portfolio management evaluation. Critiques of the Norwegian oil fund (SPU) have argued that SPU should halt all active portfolio management and instead aim for a purely passive investment strategy. In response to this critique SPU have argued that their investment strategy has delivered a return which is 0.25% larger than the return on the benchmark index set by the fund's owner (i.e., the Ministry of Finance). Moreover, they argue that the Sharpe ratio of the return on the fund is larger than the Sharpe ratio on assets in the market.

   (a) Discuss the statement: *"Investors should go for passive investments rather than active portfolio management because active management is more expensive but yields the same average rate of return as passive management."*

      - **This is the so-called "Shape's arithmetic of active management". If there are no assets being added or subtracted from the market portfolio, then it must be the case that before costs, the return on the average actively managed dollar will equal the return on the average passively managed dollar. However, after costs, the return on the average actively managed dollar will be less. These assertions will hold for any time period. Moreover, they depend only on the laws of addition, subtraction, multiplication and division. Note, however, that when the composition of the market portfolio changes, then active portfolio managers could on average beat the passive investors provided that they are better at timing the changes in the portfolio compositions.**

   (b) Explain why a positive excess return is not necessarily evidence of superior investment skills.

      - **The fund manager could use the leeway allowed by the Tracking Error to take on systematic risk, for example by choosing a portfolio with a higher market β_M of target known "risk factors" that give excess return relative to CAPM (small cap, value, etc.). This would give an expected return higher than the market return. However, it would not be evidence of superior "skill" because the premium would vanish once one risk adjusts the excess return (by for example applying a standard factor model).**

   (c) Suppose you, as a portfolio manager, is being evaluated with the measure "get as much excess return as possible given a certain Tracking

Error bound" and assume that CAPM is true. How could you beat the expectations of the fund owner?

- **Following 4b above, one should use the leeway to increase the market β and to expose oneself to known risk factors. This should be done in such a way so as to achieve the maximum expected excess return per unit of tracking error.**

(d) Explain why the Sharpe ratio of the portfolio relative to the Sharpe ratio of the market is a useful measure to evaluate SPU, provided that CAPM is true.

- **The Sharpe ratio is defined as the ratio $E(R)/\sigma(R)$, i.e., the return to variability or the return per unit of risk. If the portfolio manager can achieve a higher Sharpe ratio than the Sharpe ratio on the market portfolio, then he/she delivers an outcome strictly above the capital market line in the mean-standard deviation space. If CAPM is true that this is the only aspect of return that matters.**

(e) Define the "Appraisal ratio" for a portfolio manager who holds a portfolio p as

$$AR_p \equiv \frac{\alpha_p}{\sigma(\tilde{\varepsilon}_p)},$$

where α_p and $\tilde{\varepsilon}_p$ are derived from running the following regression:

$$\tilde{R}_p - r_f = \alpha_p + \beta_p \left(\tilde{R}_p - \tilde{R}_B \right) + \tilde{\varepsilon}_p,$$

and where $\tilde{R}_p$ and $\tilde{R}_B$ are the realized returns on the portfolio p and the benchmark portfolio, respectively, and β_p is the market beta on the portfolio p.

i. Suppose you, as a portfolio manager, is being evaluated with the AR measure, given a certain Tracking Error bound. Suppose the "pricing anomalies" of the past will persist in the future. How could you beat the expectations of the fund owner?

   - **One should use the leeway to achieve exposure to known risk factors. This would imply targeting portfolios with more stocks in small firms ("small cap"), stocks with higher book value of capital relative to the price ("value"), and stock which recently gained in price ("momentum"), Note that targeting stocks with higher market β is of no use since the market β exposure is being controlled for in the Appraisal ratio.**

ii. How would you evaluate a portfolio manager in the presence of "known" pricing anomalies?

- **One should first determine the index the portfolio manager should be measured up to. Then one should run a standard factor regression, for example a Fama-French 3-factor model – possibly adjusted for investability. The portfolio manager should ideally have a positive risk-adjusted excess return. Note, however, that being able to deliver an exposure to risk factors but with zero excess return (after risk adjustment) can be an achievement provided that this risk factor exposure is something desired by the fund owner.**

5. Smaller questions:

   (a) You observed the following situation

Security	Beta	Expected return
Renewable Energy Corp	1.3	0.23
Statoil	0.6	0.13

   Assume these securities are correctly priced. Based on the CAPM, what is the expected return on the market? What is the risk free rate?

   - **CAPM implies**

   $$E(\tilde{R}_{Statoil}) - r_f = 0.23 - r_f = 1.3(E(\tilde{R}_M) - r_f)$$
   $$E(\tilde{R}_{REC}) - r_f = 0.13 - r_f = 0.6(E(\tilde{R}_M) - r_f)$$

   Solving this linear system with two unknowns yields

   $$E(\tilde{R}_M) = 0.187\,14$$
   $$r_f = 0.04429$$

   (b) Assume that the annual risk-free rate is $r_f = 1\%$ and that the annual mean and standard deviation of the return on the Oslo Børs OBX index are 6% and 10%, respectively. Using stock prices over the last three years you find that the return on Telenor stocks have had an annualized covariance with the OBX index of $cov\left(\tilde{R}_i, \tilde{R}_M\right) = 0.006$. A share in Entra is currently priced at 100 NOK per share.

      i. Calculate an estimate of the market β of Entra.

      - **THIS EXERCISE CONTAINED AN ERROR. IT MEANT TO SAY THAT "ENTRA STOCKS HAVE HAD AN ANNUALIZED COVARIANCE WITH THE obx INDEX OF $cov\left(\tilde{R}_{Entra}, \tilde{R}_M\right) = 0.006$." In this case the β of Entra could be estimated as**

      $$\beta_{Entra} = \frac{cov\left(\tilde{R}_{Entra}, \tilde{R}_M\right)}{\sigma_M^2} = \frac{0.006}{(0.1)^2} = 0.6$$

ii. Suppose CAPM is true. What is the expected price of a Entra stock one year from now?

- **Apply CAPM:**

$$E(\tilde{R}_{Entra}) - 1\% = 0.6 \cdot (6\% - 1\%) = 3\%,$$

so the expected price in one year is $E\left(P_{Entra}\right) = 100 \cdot (1\% + 3\%) = 104$**.**

(c) Assume that every asset has the same expected return. Furthermore, all assets have the same covariance with each other. As the number of assets in the portfolio grows, which becomes more important: Variance or covariance? Why?

- **The covariances become more important. In the case when all assets have the same covariance with each other (and the same return), the optimal portfolio must be equally weighted. Then as the number of assets go to infinity, the variance of the portfolio will converge to the covariance between assets and the variance of each individual asset becomes irrelevant.**