Final Exam, Make Up

Question 1: Resource Extraction (50%).
Write down the objective of a representative (competitive) firm that extracts oil. The extraction costs should increase in the amount already extracted (as reservoirs shrink extraction becomes more expensive). You can decide whether to use general functional forms or a (reasonable) example. Use continuous time, set up the Hamiltonian, and derive the Euler equation. Interpret the Euler equation. Relate your result to Hotelling's rule.

Solution:

The solution expects:

- An objective function and constraint of the form

$$\max \int_0^T (p_t q_t - C(q_t, R_t))e^{-rt}\mathrm{d}t \quad \text{s.t. } \dot{R} = -q_t \text{ , } R_0 = \bar{R} \text{ , } R_t \geq 0.$$

  with $C'_{q_t} > 0$ and $C'_{R_t} < 0$. Or similar (e.g. particular functional forms).

- The current value Hamiltonian is

$$H^c(q_t, \mu_t) = p_t q_t - C(q_t, R_t) + \mu_t(-q_t)$$

  or, alternatively, present value Hamiltonian.

- The necessary conditions

$$\frac{\partial H^c(q_t, \mu_t)}{\partial q_t} = p_t - C'_{q_t} - \mu_t \stackrel{!}{=} 0$$
$$\frac{\partial H^c(q_t, \mu_t)}{\partial R_t} = -C'_{R_t} \stackrel{!}{=} r\mu_t - \dot{\mu}_t$$

- The corresponding Euler equation, here

$$\frac{\frac{d}{dt}(p_t - C'_{q_t})}{p_t - C'_{q_t}} = r - \frac{-C'_{R_t}}{p_t - C'_{q_t}}$$

- An interpretation of the Euler equation:
  Net marginal revenue has to increase at the rate of interest less the extraction cost increase that is caused by a decline of the available resource in the ground (relative to net marginal revenue). The last term is a result of a particular benefit from leaving a unit of the resource in the ground: extracting the other units while that additional unit of the resource remains in the ground will be slightly cheaper because it reduces the increasing extraction costs. That benefit reduces the required opportunity costs of leaving a unit in the ground.
  This latter reasoning can be stated as part of the interpretation of the Euler equation or in connected to the relation to Hotelling's rule.

- In Hotelling's rule the price of the resource has to increase at the rate of interest. That result holds if the firm does not face extraction costs. Here, instead, the *net marginal revenue* has to grow at the rate of interest (or less, see below) because only the difference between price and marginal cost (net marginal revenue) is subject to the opportunity investment at market interest. In addition, the net marginal revenue does not have to grow at the rate of interest because leaving an additional unit of the resource in the ground from one period to the next has the additional payoff that extracting other units during that period becomes slightly cheaper.

Question 2: Phase Diagram and Tipping Points (50%).
Lakewood county's lake used to be known for its clear water, a high amount of fish, and many recreational visitors. Over the last decade, agricultural usage of the area expanded and part of the fertilizer was carried by the watershed into the lake. While the locals noted a slow increase in algae over the last years, this summer the lake suddenly 'turned green' from one month to the next. The dynamics of the phosphorous stock in the lake causing the growth of algae is governed by the equation

$$\dot{x}_t = c_t - \alpha x_t + f(x) \tag{1}$$

where c denotes the phosphorous inflow. You will not need the precise functional form of f (it is $f(x) = \frac{x^2}{1+x^2}$ but using "f" is fine for calculations and the relevant graphs are provided below). Figure 1 depicts the $\dot{x} = 0$ curve.

i) What is α? What does the function f reflect? Use Figure 1 and the $\dot{x} = 0$ curve to explain what happened in Lakewood county. For this purpose, copy the Figure into your worksheet and illustrate your reasoning with arrows indicating the lake dynamics. Mark an exemplary point indicating the state of the lake before and a point indicating the state of the lake after the summer.

Solution: α denotes the self purification rate of the lake (or decay rate of the pollutant). The function f characterizes a non-linear (convex-concave) feedback: at a certain point, as additional phosphor enters the lake, some of the phosphor initially stored in the plants and the soil will be released (in particular as a result of the algae growth and decay suffocating the other plants) so that the lake's increase in phosphorous levels is faster than the inflow. Before the summer the state of the lake was to the left and below the first hump (on or slightly above the curve). After the summer, the lake's state is on or around the rightermost arm of the curve on a segment above or in the area of the height of the hump. The arrows indicating the movement take it from the first state to the second once phosphorous inflow exceeds the height of the hump.

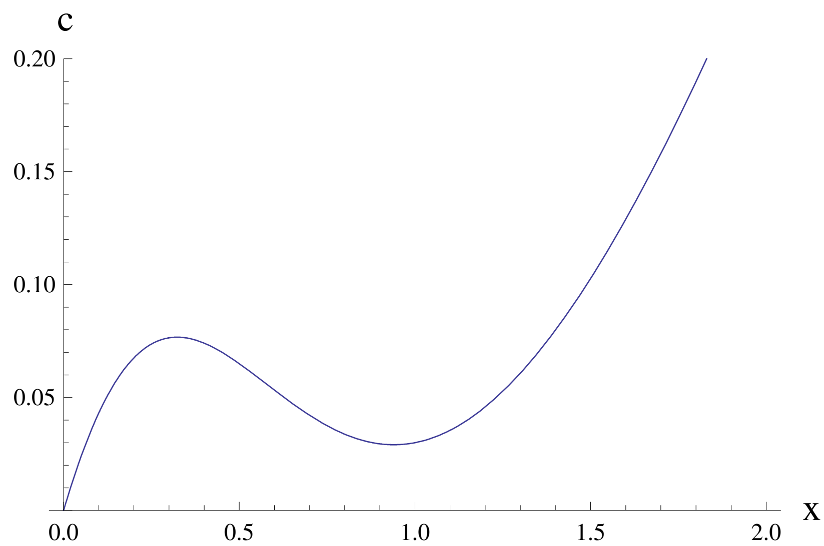

Figure 1: $\dot{x}_t = 0$ curve for Lakewood's lake.

A social planner is now charged to optimally control the lake. She faces benefits from consumption that are normalized to coincide with the phosphorous flow c_t and she has to trade-off these benefits against pollution costs as the pollutant stock x_t accumulates. She maximizes

$$W = \int_0^\infty (\ln c - \beta x^2) e^{-\rho t} dt \quad \text{subject to equation (1) and } x_0 = \bar{x}.$$

ii) Set up the current value Hamiltonian and derive the Euler equation.

Solution: The current value Hamiltonian is

$$H(c, x, \mu) = \ln c - \beta x^2 + \mu(c - \alpha x + f(x))$$

delivering the first order conditions

$$\frac{\partial H(c,x,\mu)}{\partial c_t} = \frac{1}{c_t} + \mu_t \overset{!}{=} 0$$
$$\Rightarrow \mu_t = -\frac{1}{c_t}$$
$$\Rightarrow \dot{\mu}_t = \frac{\dot{c}_t}{c_t^2}$$
$$\frac{\partial H(c,x,\mu)}{\partial x_t} = -2\beta x_t - \mu_t(\alpha + f'(x_t)) \overset{!}{=} \mu_t \rho - \dot{\mu}_t$$
$$\Rightarrow \dot{\mu}_t = \mu_t(\rho + \alpha - f'(x_t)) + 2\beta x_t \ .$$

Inserting the expressions for μ_t and $\dot{\mu}_t$ delivers the Euler equation

$$\frac{\dot{c}_t}{c_t^2} = -\frac{1}{c_t}(\rho + \alpha - f'(x_t)) + 2\beta x_t$$
$$\Leftrightarrow \dot{c} = 2\beta x c^2 - (\alpha + \rho - f'(x))\, c$$

Figure 2 adds the corresponding $\dot{c}_t = 0$ line into the phase diagram.

iii) Copy the Figure qualitatively into your notes. Draw arrows indicating the qualitative dynamics of c_t and x_t into every region of the diagram (make sure to do it for all regions). Derive the direction of the qualitative arrows formally from the $\dot{x}_t$ and the $\dot{c}_t$ equations, explaining what you are doing.
Hint: It is helpful to calculate the $\dot{c}_t = 0$ line and use it in evaluating one of the conditions.

Solution: Qualitative dynamics of x_t relative to $\dot{x}_t = 0$ line:

$$\dot{x}_t = c_t - \alpha x_t + f(x_t) \equiv G(x_t, c_t)$$
$$\frac{\partial G(x_t, c_t)}{\partial c_t} = 1$$

Hence, above the $\dot{x}_t = 0$ line the stock is increasing and below it is decreasing.

Qualitative dynamics of c_t relative to $\dot{c}_t = 0$ line:

$$\dot{c} = 2\beta x c^2 - (\alpha + \rho - f'(x))\, c \equiv C(c, x)\,.$$

On the $\dot{c} = 0$ line we have

$$c = 0 \quad \text{or} \quad c = \frac{\alpha + \rho - f'(x)}{2\beta x} \equiv g(x) \tag{2}$$
$$\Rightarrow \alpha + \rho - f'(x) = 2\beta x c \quad \text{for} \quad c > 0\,. \tag{3}$$

Given logarithmic utility from loading, $c = 0$ will not be an economically reasonable solution. It is easiest is to check the dynamics above and below the $\dot{c} = 0$ curve using equation (3):

$$\left.\frac{\partial C(c_t, x_t)}{\partial c_t}\right|_{\dot{c}=0} = -(\alpha + \rho - f'(x)) + 4\beta x c = 2\beta x c > 0 \tag{4}$$

where we used equation (3) assuming $c > 0$.
Thus, the optimal loading increases above the $\dot{c} = 0$-line and decreases below.

iv) Qualitatively, draw the optimal control trajectories (separatrix) into the figure. Is there a Skiba point? If so, mark it. Briefly discuss the qualitative features of the optimal dynamics assuming that the starting point is the "tipped" laked.

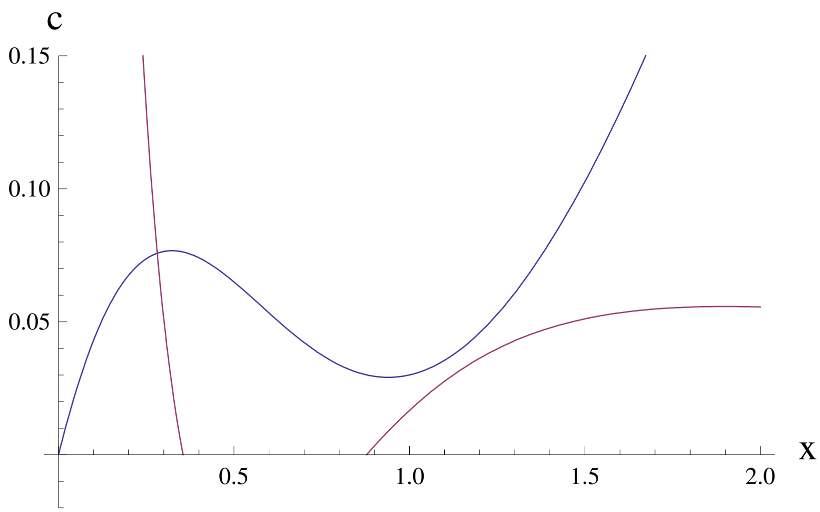

Figure 2: Phase diagram: $\dot{x}_t = 0$ and $\dot{c}_t = 0$ lines.

Solution: No, there is no Skiba point. In this problem there is a unique optimal steady state independent of the initial pollution level.

Dynamics: From the left the separatrix falls into the steady state. It continues to fall on the other side of the steady state and eventually crosses the $\dot{c}_t = 0$ line starting to increase. Or, following the direction of the arrows, to the very right of the diagram it will start below the $\dot{c}_t = 0$ curve falling until it crosses over to the left of the $\dot{c}_t = 0$ from where it starts to increase into the steady state. Thus, starting from the "tipped" lake, the pollution inflow first has to fall underneath the dip of the $\dot{x}_t = 0$ curve before it can increase again approaching the steady state at the intersection of the $\dot{x}_t = 0$ and $\dot{c}_t = 0$ line.