Final Exam – Guidelines

Resource Economics - ECON4925
Department of Economics, University of Oslo

December 14^{th}, 2022

This exam has 3 questions. The points of each question x are mentioned in brackets as [x p]. The exam sums to a total of 100 points. Provide well-motivated answers that are as concise as possible. If you get stuck in algebra or discover a mistake, explain how you would proceed, what you would expect, or why you think there is a mistake.

1. In this question, choose the right answer – "True" or "False" for part i, and "True", "False" or "It Depends" for part ii. [30 p]

   i. True or false? The Faustmann rotation [10 p]

      (a) considers both logging and non-logging benefits of harvests in forestry. [2 p]

      (b) gives the optimal rotation length of a model with one hundred rotations. [2 p]

      (c) assumes that operation costs within a rotation increase the longer said rotation is. [2 p]

      (d) is shorter than the rotation that would maximize the mean annual increment. [2 p]

      (e) is longer than the optimal single rotation. [2 p]

      Solution: a)F b)F c)F d)T e)F.

   ii. True or false or it depends? If you choose "it depends" then you need to provide an explanation on what factor(s) and how it depends. Keep the answers concise.

      Assume a finite supply of a non-renewable resource. Consider the following scenarios regarding the optimal extraction plan. The optimal time horizon could be either finite or infinite. True, false or it depends? [20p]

      (a) If firms are competitive, it is not possible to decentralize resource extraction such that the outcome is socially optimal. [10 p]

      (b) A social planner with constant absolute risk aversion ($\frac{1-e^{-at}}{a} + b, a > 0, b \in \mathbb{R}$) will optimally choose to extract over a finite time horizon, and will exhaust the resource stock fully. [10 p]

      Solution: a) It depends on whether or not there are per period fixed costs in production. If there are no fixed costs, then there is a socially optimal decentralization. If there are fixed costs, however, the social planner terminates at a lower resource flow than competitive firms because of consumer benefit. Hence, in this case, there is no socially optimal decentralization. b) It depends on the value of b. If $b = 0$, the planner chooses to extract over a finite horizon and exhausts the stock fully. If $b > 0$, $U(0) > 0$ and thus the planner chooses an infinite time horizon, but will run out of the resource at the same point in time as when $b = 0$. If $b < 0$, the planner chooses a finite time horizon, but will stop at a strictly positive flow and utility level (i.e. will not exhaust the stock fully).

2. **Fishery Economics:** Consider a fishery where the biomass of some fish S_t in international waters evolves according to the equation of motion

$$\dot{S}_t = g(S_t) - H_t,$$

where time is continuous, $H_t \geq 0$ is the size of harvest, and

$$g(S_t) = r\left(2\sqrt{S_t} - \frac{S_t}{\sqrt{M}}\right),$$

gives the dynamics of the fish population in the absence of harvesting, where $r > 0$ and $M > 0$ are parameters defined by the biological conditions of the marine ecosystem. Initial fish stock is positive $S_0 > 0$. (Observe that the derivative of the function $f(x) = \sqrt{x} = x^{0.5}$ is $f'(x) = \frac{1}{2\sqrt{x}}$.) [40p]

(a) Find the maximum sustainable yield S^{MSY}. [3p]

$g'(S_t) = 0 \Rightarrow r\left(\frac{1}{\sqrt{S_t}} - \frac{1}{\sqrt{M}}\right) = 0 \Rightarrow S^{MSY} = M.$

(b) Find the carrying capacity of the fish population S^{CC}. [3p]

$g(S_t) = 0 \Rightarrow r\left(2\sqrt{S_t} - \frac{S_t}{\sqrt{M}}\right) = 0 \Rightarrow 2\sqrt{S_t} - \frac{S_t}{\sqrt{M}} \Rightarrow 4S_t = \frac{S_t^2}{M} \Rightarrow S_t^2 - 4MS_t = 0 \Rightarrow S_t - 4M = 0 \Rightarrow S^{CC} = 4M$ (and $g(0) = 0$).

A social planner (a state or government) will choose the optimal sizes of harvest H_t over an infinite time horizon, with immediate utility $u(H_t) = 2\sqrt{H_t}$, and discounts the utility stream at the an interest rate $i \geq 0$.

(c) Formulate the dynamic optimization problem that the social planner faces, and identify the control and state variables. [5p]

$$\max_{H_t \geq 0} \int_0^\infty 2\sqrt{H_t} e^{-it} dt$$

$$\text{such that } \dot{S}_t = r\left(2\sqrt{S_t} - \frac{S_t}{\sqrt{M}}\right) - H_t$$

and given $S_0 > 0$.

The state variable is the biomass of the fish (fish stock) S_t and the control variable is the the size of harvest H_t.

(d) Formulate the current-value Hamiltonian and derive the first-order conditions. [5p]

Current-value Hamiltonian $\mathcal{H}^c(S_t, H_t, \mu_t) = 2\sqrt{H_t} + \mu_t[g(S_t) - H_t]$. The first-order conditions (FOC):

1. $\frac{\partial \mathcal{H}^c}{\partial H_t} = 0 \Rightarrow \frac{1}{\sqrt{H_t}} - \mu_t = 0 \Rightarrow \mu_t = \frac{1}{\sqrt{H_t}}$ or $H_t = \frac{1}{\mu_t^2}$;
2. $\frac{\partial \mathcal{H}^c}{\partial S_t} = i\mu_t - \dot{\mu}_t \Rightarrow \mu_t g'(S_t) = i\mu_t - \dot{\mu}_t \Rightarrow r\left(\frac{1}{\sqrt{S_t}} - \frac{1}{\sqrt{M}}\right) = i - \frac{\dot{\mu}_t}{\mu_t}$;
3. $\frac{\partial \mathcal{H}^c}{\partial \mu_t} = \dot{S}_t \Rightarrow \dot{S}_t = g(S_t) - H_t,$ i.e. the equation of motion.

(e) Show that the Euler equation is given by [4p]

$$\frac{\dot{H}_t}{H_t} = 2\left(r\left[\frac{1}{\sqrt{S_t}} - \frac{1}{\sqrt{M}}\right] - i\right).$$

From the first FOC we know that

$$H_t = \frac{1}{\mu_t^2} \xrightarrow[\text{respect to time}]{\text{Differentiation with}} \dot{H}_t = \frac{-2\dot{\mu}_t \mu_t}{\mu_t^4} = \frac{-2\dot{\mu}_t}{\mu_t} \cdot \underbrace{\frac{1}{\mu_t^2}}_{=H_t} \Rightarrow \frac{\dot{\mu}_t}{\mu_t} = -\frac{1}{2} \cdot \frac{\dot{H}_t}{H_t}.$$

Replacing the expression for $\frac{\dot{\mu}_t}{\mu_t}$ in the second FOC with the expression above, we get

$$r\left(\frac{1}{\sqrt{S_t}} - \frac{1}{\sqrt{M}}\right) = i + \frac{1}{2} \cdot \frac{\dot{H}_t}{H_t},$$

and rearranging we get the Euler equation as in the question.

(f) Find the optimal steady state level of fish stock S^{stst} resulting from this optimization problem. [4p]

The steady state level of fish stock S^{stst} is a point S^{stst} where there no longer is change in the dynamic system, i.e. $\dot{H}_t = 0$ and $\dot{S}_t = 0$. The latter, $\dot{S}_t = 0$ yields $H_t = g(S_t)$, and thus will not directly yield S^{stst}. On the other hand, $\dot{H}_t = 0$ yields $H_t = 0$ or

$$2\left(r\left[\frac{1}{\sqrt{S_t}} - \frac{1}{\sqrt{M}}\right] - i\right) = 0 \Rightarrow \sqrt{S_t} = \frac{1}{\frac{i}{r} + \frac{1}{\sqrt{M}}}$$

$$\Rightarrow S^{stst} = \frac{1}{\left(\frac{i}{r} + \frac{1}{\sqrt{M}}\right)^2} = \frac{M}{\left(\frac{i}{r}\sqrt{M} + 1\right)^2},$$

where the last equation is given by multiplying the numerator and denominator with M. Observe that there are two other stedy states given by where $H_t = 0$. The corresponding fish stock levels at these steady states, are $S = 0$ and $S = S^{CC} = 4M$.

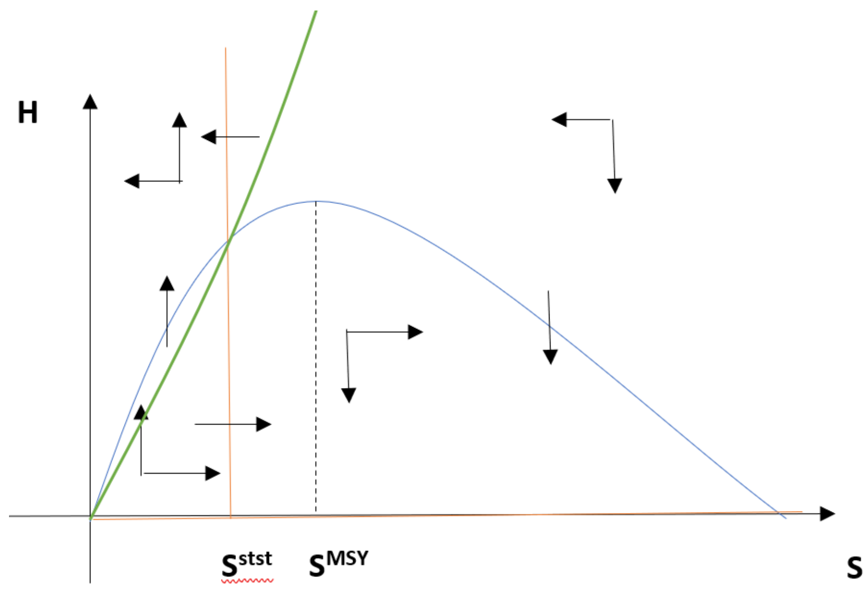
(g) Under which circumstances is the steady state level of fish stock equal to the maximum sustainable yield ($S^{stst} = S^{MSY}$)? Is one larger than the other in general? If yes, which? Explain. [6p]

From (h) we have

$$S^{stst} = \frac{M}{\left(\frac{i}{r}\sqrt{M} + 1\right)^2},$$

which is equal to $S^{MSY} = M$ if $i = 0$. In other words, in the absence of discounting the social planner chooses the maximum sustainable yield as the optimal steady state, since the opportunity cost of postponing capture is zero. When $i > 0$, then there is an opportunity cost to postponing capture, and as a result fish are caught more frequently, and thus $S^{stst} \leq S^{MSY}$ in general.

(h) Use the equation of motion for the fish stock and the Euler equation to perform phase plane analysis on the solutions to the problem. Put S_t and H_t on the horizontal and vertical axes respectively. Illustrate the separatrix, and explain what it represents economically. [10p]



The blue curve gives the nullcline $\dot{S} = 0$, and the red curves that of $\dot{H}_t = 0$. The planner optimizing over infinite horizon would steer the harvesting path along the separatrix (the green line) towards the only economically viable steady state $(S, H) = (S^{stst}, g(S^{stst}))$, along the unique separatrix (given that this steady state is a saddle point). If $S_0 > S^{stst}$, then the planner would decrease harvest over time towards $H = g(S^{stst})$. If $S_0 < S^{stst}$, then the planner would increase harvest over time towards $H = g(S^{stst})$. Finally, if $S_0 = S^{stst}$, then the planner would keep harvest constant over time at $H = g(S^{stst})$.

3. **Lake Pollution:** A social planner wants to feed their population. Agricultural activity at each point in continuous time t, however, will entail polluting a nearby lake by a phosphorous loading c_t, which will increase the stock of phosphorous x_t. The planner, hence, faces the following welfare maximization problem:

$$\max_{c_t \geq 0} \int_0^\infty [u(c_t) - \beta x_t^2] e^{-\rho t} dt,$$
$$\text{such that } \dot{x}_t = c_t - \alpha x_t + f(x_t),$$

where $u'(c) > 0$ and $u''(c) < 0$, and initial stock of pollution $x_0 > 0$ is given. Moreover, $f(x) = \frac{x^2}{1+x^2}$. (Observe that $f'(x) = \frac{2x}{(1+x^2)^2}$, $f''(x) = \frac{2-6x^2}{(1+x^2)^3}$.) Moreover, $0 < \alpha < 1$, $\rho > 0$ and $0 < \beta < 1$. [30p]

(a) Explain the concept of a Skiba point in a resource management problem. What differs in the underlying dynamic system as opposed to typical dynamic systems that do not contain Skiba points? What are the implications for the solution set? Explain why it may emerge in the context of the optimization problem above. (No equations or graphs are required, but provide any that would help in explaining the concept.) [20p]

The Skiba point $x^s > 0$ in this context, is the level of initial pollution stock $x_0 = x^s$ such that the planner is indifferent between several potentially optimal solution trajectories (policy or decision plans). Typically, for Skiba points there are multiple bioeconomic equilibria (steady states) that emerge in the phase portrait of the decision rules. In the context of lake pollution, this means at least three equilibria. Two saddle points, one with lower steady-state level of pollution x^c (clean lake steady state), and one with higher level of steady-state pollution stock x^d (dirty lake steady state) – i.e. $x^c < x^d$. The third equilibria is repulsive (unstable) and lies in between the other two $x^c < x^u < x^d$. Also, $x^c < x^s < x^d$. If $x_0 > x^s$, then the optimal solution path is a unique one leading to the dirty steady state, whereas if $x_0 < x^s$ then the optimal solution path is a unique one leading to the clean steady state. The reason why Skiba points emerge, is nonconvexities in the Hamiltonian. Here the nonconvexity is due to the function $f(x)$ modeling the pollution interaction in the lake. The function f makes the system non-convex, since f'' is not negative everywhere. Indeed, it is positive for small values of the pollution stock x_t, modeling stronger feedback of phosphorous for low levels of pollution stock. At these levels, there is namely more room for algae to grow. As a result of such non-convexity, the system can have multiple trajectories that satisfy all the necessary conditions including transversality. These trajectories can converge into different steady states. The necessary conditions of the maximum principle are generally not enough in order to pin down the optimal control for any state of the system.

The function $u(c_t)$ represents the immediate utility stream from agricultural activity. The planner, however, is unsure of the exact benefits of the farming products for the population. It is therefore considering two utility functions: $u_1(c_t) = \log c_t$ and $u_2(c_t) = \frac{1-e^{-ac_t}}{a}, a > 0$.

(b) Show that if $u(c_t) = u_2(c_t)$, then $c_t = 0$ when $\mu_t \leq -1$, where μ_t is the current-value shadow value of pollution stock. What is the corresponding condition for $u(c_t) = u_1(c_t)$? [6p]

The current value Hamiltonian is given by:

$$\mathcal{H}^c(c_t, x_t, \mu_t) = u_2(c_t) - \beta x_t^2 + \mu_t[c_t - \alpha x_t + f(x_t)]$$

The first FOC is given by:

$$\frac{\partial \mathcal{H}^c}{\partial c_t} = 0 \Rightarrow e^{-ac_t} + \mu_t = 0 \Rightarrow c_t = -\frac{1}{a}\log(-\mu_t).$$

Hence $c_t = 0$, when $\mu_t = -1$. Moreover, $\frac{\partial c_t}{\partial \mu_t} = -\frac{1}{a\mu_t} > 0$ ($\mu_t < 0$ as pollution stock is a bad and thus has negative shadow value). Hence, $c_t = 0$, whenever $\mu_t \leq -1$. (Alternatively, $c_t > 0 \Rightarrow -\frac{1}{a}\log(-\mu_t) > 0 \Rightarrow \log(-\mu_t) < 0 \Rightarrow -\mu_t < 1 \Rightarrow \mu_t > -1$. Thus, $c_t = 0$ in all other cases, i.e. $\mu_t \leq -1$.)
There is no corresponding condition for the the first case scenario, since when $u(c_t) = u_1(c_t) = \log c_t$, a choice of no agricultural activity $c_t = 0$ will never be optimal. This is seen in two different ways. First, $\lim_{c_t \to 0} \log c_t = -\infty$, i.e. the population enters the worst case scenario of indefinitely large disutility. Second, by looking at the marginal utility $u_1'(c_t) = \frac{1}{c_t}$ we can see that $\lim_{c_t \to 0} u_1'(c_t) = \infty$. In other words, the population is infinitely better off if the planner allows for marginally higher agricultural production.

(c) Which of these two utility structures is more likely to induce a Skiba point for a given parameter space? Why? Motivate your answer concisely. [4p]

A Skiba point is more likely in the first scenario. As indicated in part (d), when $u(c_t) = u_1(c_t) = \log c_t$, a choice of no agricultural activity $c_t = 0$ will never be optimal. However, $c_t = 0$ optimally when $\mu_t \leq -1$ in the second scenario, since the marginal utility at zero activity $u_2'(0) = 1$ is finite. Given that having no production is a viable optimal choice in the second scenario, the parameter space yielding Skiba points is likely smaller than in the first scenario. In other words, by valuing production lower than the first scenario, in the second scenario the likelihood of the parameter space inducing a dirty-lake steady state, and an optimal path leading to it, diminishes.