FIL2405/4405 Philosophical Logic and the Philosophy of Mathematics, Spring 2018 Exam

Wednesday 16 May 2018, 14:30–18:30 (4 hours)

The exam consists of 3 pages. No aids (*hjelpemidler*) are allowed, and answers must be in English.

There are 26 questions, which are weighted equally. They are divided into two parts, the first containing 18 questions on propositional modal logic and the second containing 8 questions on quantified modal logic. **Answer 9 questions from the first part, and 4 questions from the second part.** If you have time, you may answer more than 9 questions from the first part and 4 questions from the second part; in this case, the 9 questions of the first part answered best and the 4 questions of the second part answered best will be counted.

Part 1: Propositional Modal Logic.

Let $\mathrm{D_c}$ be the formula $Mp \supset Lp$. Let C be the class of frames $\langle W, R \rangle$ which are such that for each world $w \in W$, there is at most one world $v \in W$ such that wRv. Note that these are the frames in which for all worlds w, v, u, if wRv and wRu, then $v = u$.

(a) Derive E in the system $\mathrm{K} + \mathrm{T} + \mathrm{D_c}$. (E is $Mp \supset LMp$ and T is $Lp \supset p$.)

(b) Discuss whether $\mathrm{D_c}$ is a plausible principle on the temporal interpretation which interprets L as 'at every future point in time'.

(c) Discuss whether $\mathrm{D_c}$ is a plausible principle on the temporal interpretation which interprets L as 'in precisely 5 minutes'.

(d) Show that $\mathrm{D_c}$ is valid on all frames in C (defined above).

(e) Show that $\mathrm{D_c}$ is not valid on any frame not in C.

(f) Explain how it follows from (d) that $\mathrm{K} + \mathrm{D_c}$ is sound with respect to C.

(g) Show that it is not the case that D is valid on all frames in C. (D is $Lp \supset Mp$.)

(h) Explain how it follows from (f) and (g) that D cannot be derived in $\mathrm{K}+\mathrm{D_c}$.

(i) Show that the canonical frame for $\mathrm{K} + \mathrm{D_c}$ is in C.

(The canonical frame for $\mathrm{K} + \mathrm{D_c}$ is the frame $\langle W, R \rangle$ where W is the set of maximal $\mathrm{K} + \mathrm{D_c}$-consistent sets of formulas, and R is the relation on W such that wRv iff for every formula α, if $L\alpha \in w$ then $\alpha \in v$.)

(j) Explain how it follows from (i) that $\mathrm{K}+\mathrm{D_c}$ is complete with respect to C.

(k) Show that $(Mp \wedge Mq) \supset M(p \wedge q)$ is valid on all frames in C by reasoning about truth in worlds of models based on frames in C.

(l) Explain how it follows from (j) and (k) that $(Mp \wedge Mq) \supset M(p \wedge q)$ is derivable in $\mathrm{K} + \mathrm{D_c}$.

(m) Derive $(Mp \wedge Lq) \supset Mq$ in K.

(n) Confirm, using (m) if needed, that the conclusion arrived at in (l) is correct by deriving the formula $(Mp \wedge Mq) \supset M(p \wedge q)$ in the system $\mathrm{K} + \mathrm{D_c}$.

(o) Derive $\mathrm{D_c}$ in the system $\mathrm{K} + (Mp \wedge Mq) \supset M(p \wedge q)$.

(p) Explain how it follows from (n) and (o) that $\mathrm{K}+\mathrm{D_c}$ and $\mathrm{K}+(Mp \wedge Mq) \supset M(p \wedge q)$ have the same theorems.

(q) Show that the following is a derived rule of $\mathrm{K} + \mathrm{D_c}$:

$$\vdash \alpha \supset \beta \rightarrow \vdash M\alpha \supset L\beta$$

(r) Show, using (h) if needed, that the following is not a derived rule of $\mathrm{K}+\mathrm{D_c}$:

$$\vdash \alpha \supset \beta \rightarrow \vdash L\alpha \supset M\beta$$

Part 2: Quantified Modal Logic.

(s) Choose a particular instance of the Barcan formula and a particular informal interpretation of the predicates occurring in it, and use this as an example in explaining why some philosophers have doubted that the Barcan formula should be included in systems of quantified modal logic.

(The Barcan formula is the following schema, where x may be any variable and α may be any formula:

BF : $\forall x L\alpha \supset L\forall x\alpha$)

(t) Choose a particular instance of the converse Barcan formula and a particular informal interpretation of the predicates occurring in it, and use this as an example in explaining why some philosophers have doubted that the converse Barcan formula should be included in systems of quantified modal logic.

(The converse Barcan formula is the following schema, where x may be any variable and α may be any formula:

BFC : $L\forall x\alpha \supset \forall x L\alpha$)

(u) For an arbitrary given formula α, derive $\forall x\forall y\alpha \supset \forall y\forall x\alpha$ in LPC + K.

(For any system S of modal propositional logic, the system LPC + S of modal predicate logic is defined as follows:

S′ : If α is an LPC substitution-instance of a theorem of S then α is an axiom of LPC + S.

∀1 : If α is any wff and x and y any variables and $\alpha[y/x]$ is α with free y replacing every free x, then $\forall x\alpha \supset \alpha[y/x]$ is an axiom of LPC + S.

N : If α is a theorem of LPC + S then so is $L\alpha$.

MP : If α and $\alpha \supset \beta$ are theorems of LPC + S then so is β.

∀2 : If $\alpha \supset \beta$ is a theorem of LPC + S and x is not free in α then $\alpha \supset \forall x\beta$ is a theorem of LPC + S.

You may rely on the fact that all instances of BFC are derivable in LPC + S, and on the fact that $\mathrm{UG}^{\supset}$ is a derived rule of LPC + S:

$\mathrm{UG}^{\supset}$: $\vdash \alpha \supset \beta \rightarrow \vdash \forall x\alpha \supset \forall x\beta$)

(v) Derive, using (u) if needed, $\forall x L\forall y\phi xy \supset \forall y L\forall x\phi xy$ in LPC + K + BF.

(w) Construct a variable-domain model based on a universal frame (i.e., a frame in which every world can see every world) such that the formula $\forall x L\forall y\phi xy \supset \forall y L\forall x\phi xy$ is false in some world of this model.

(x) Derive, using (q) if needed, $\exists x M\phi x \supset L\exists x\phi x$ in LPC + K + $\mathrm{D_c}$.

(y) Construct a variable-domain model based on a frame in C (defined in Part 1) such that the formula $\exists x M\phi x \supset L\exists x\phi x$ is false in some world of this model.

(z) Construct a constant-domain model such that the formula $\exists x M\phi x \supset L\exists x\phi x$ is false in some world of this model.