

AST1100 Lecture Notes

21: Quantum gases

In this lecture we will discuss a quantum phenomenon known as degeneration. A gas which is degenerate has very special properties. The helium flash was caused by a degenerate stellar core. We will now see that the final result of stellar evolution is also a star which consists of degenerate gas: a white dwarf or a neutron star. What gives the degenerate gas its peculiar properties is its *equation of state*. The equation of state is an equation relating the pressure of the gas to its density and temperature. We have already encountered two such equations of state, $P = \rho kT / (\mu m_H)$ for an ideal gas and $P = (1/3) a T^4$ for a photon gas (we will now use capital P to denote pressure in order to distinguish it from the momentum p). We will therefore start by studying how we can obtain an expression for the pressure of a gas.

1 Pressure

To calculate the pressure of a gas we need to consider the force that particles in a gas exert on a wall (real or imaginary). Pressure is defined as the force per area on the wall $P = F/A$ from the particles in the gas. In figure 1 we see particles within a cylinder of length Δx . The particles collide with the wall at the right end of the cylinder. The surface area of the part of the wall limiting the cylinder on the right end is A . We will assume that the collisions are elastic. In elastic collisions the total absolute value of the momentum of the particle is conserved. If we define the x-direction as the direction towards the wall and the y-direction as the direction along the wall (see again figure 1), then momentum p_y in y-direction is always conserved since there is no force working in that direction. For the absolute value of the momentum to be conserved, the absolute value of the momentum p_x in the x-direction must be conserved. This means that the momentum in the x-direction after the collision must be $-p_x$ where p_x is the momentum before the collision. This means simply that the incoming angle equals the outgoing angle (see figure 2).

Before the collision, a particle has momentum (p_x, p_y) . After the collision

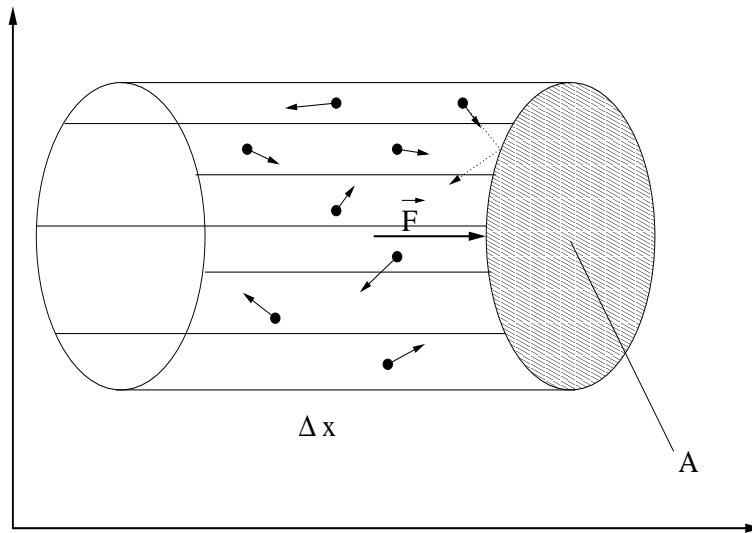


Figure 1: Pressure on the wall of area A is the total force exerted from the particles within the cylinder divided by the area.

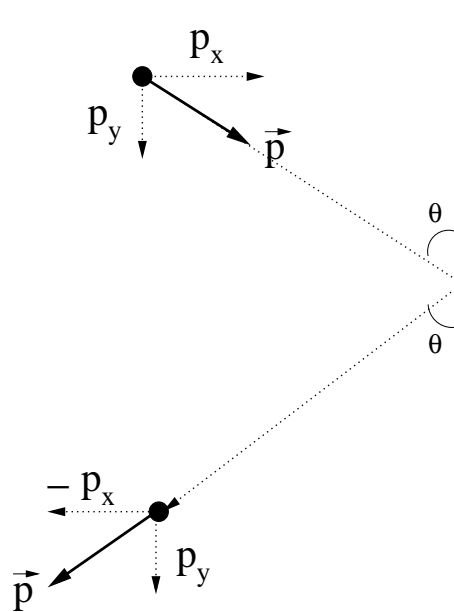


Figure 2: An elastic collision: the absolute value of the momentum is conserved.

the particle has momentum $(-p_x, p_y)$. The total change in momentum is just $2p_x$. The force exerted on the wall during a time interval Δt is according to Newton's second law

$$f = \frac{dp}{dt} \approx \frac{\Delta p}{\Delta t} = \frac{2p_x}{\Delta t}. \quad (1)$$

We will now consider a time interval Δt such that all the particles within the cylinder with velocity $|v_x|$ and momentum $|p_x|$ has collided with the wall within time Δt . The time it takes a particle with x-velocity v_x at a distance Δx from the wall to hit the wall is $\Delta t' = \Delta x/v_x$. Within the time $\Delta t'$, all the particles in the cylinder traveling in the direction towards the wall (with velocity v_x) have collided with the wall. But only half of the particles travel in the direction towards the wall. The other half travels in the opposite direction and had therefore already hit the wall within a time interval $\Delta t'$ earlier. So within the time

$$\Delta t = 2\Delta t' = \frac{2\Delta x}{v_x}$$

every single particle in the cylinder with velocity $|v_x|$ has collided with the wall. Inserting this in equation 1 we find that the force exerted by any single particle of velocity $|v_x|$ and momentum $|p_x|$ in the cylinder within the time Δt is

$$f = \frac{2p_x}{\Delta t} = \frac{v_x p_x}{\Delta x}.$$

The total velocity is given by $v^2 = v_x^2 + v_y^2 + v_z^2$, but on average the velocity components are equally distributed among all three dimensions such that $\langle v_x^2 \rangle = \langle v_y^2 \rangle = \langle v_z^2 \rangle$ giving that $v^2 = 3v_x^2$ or $v_x = v/\sqrt{3}$. Exactly the same argument holds for the momentum giving $p_x = p/\sqrt{3}$. We thus have

$$p_x v_x = \frac{1}{3} p v.$$

such that

$$f = \frac{v p}{3\Delta x}.$$

We have a distribution function $n(p)$ which gives us the number density of particles in the gas with momentum p such that $n(p)dp$ is the number of particles with momentum between p and $p + dp$. We have already seen at least one example of such a distribution function: The Maxwell-Boltzmann distribution function for an ideal gas. We used this for instance to find the

width of a spectral line as well as to find the number of particles with a certain energy in a stellar core when calculating nuclear reaction rates. We will here assume a general distribution function $n(p)$. Since $n(p)$ is a number density, i.e. number per unit volume, we can write the total particles in the cylinder $N(p)$ with momentum p as the density times the volume $A\Delta x$ of the cylinder

$$N(p) = n(p)A\Delta x.$$

The total force exerted on the wall of the cylinder by particles of momentum p is then

$$dF = \frac{pv}{3\Delta x}N(p)dp = \frac{1}{3}pv n(p) A dp,$$

or in terms of the pressure exerted by these particles

$$dP = \frac{dF}{A} = \frac{1}{3}pv n(p) dp. \quad (2)$$

We obtain the total pressure by integrating

$$P = \frac{1}{3} \int_0^\infty pv n(p) dp \quad (3)$$

which is the *pressure integral*. Given the distribution function $n(p)$ (for instance the Maxwell-Boltzmann distribution) and an expression relating v and momentum p (for instance $v = p/m$ for non-relativistic particles) we can integrate this equation to obtain the pressure in the gas.

2 Distribution functions

A statistical distribution function $n(p)$ describes how the momenta are distributed between the particles in a gas. It tells us the number density of particles having a specific momentum p . The density of particles with momentum between p and $p + dp$ is given by $n(p)dp$. By making substitutions (for instance $p = mv$), we can obtain the velocity distribution function $n(v)$ which we used to obtain the width of a spectral line in the lectures on electromagnetic radiation. Or by making the substitution $E = p^2/(2m)$ we can obtain the distribution function $n(E)$ giving the number density of particles having a certain energy E . We used the latter in the lecture on nuclear reactions.

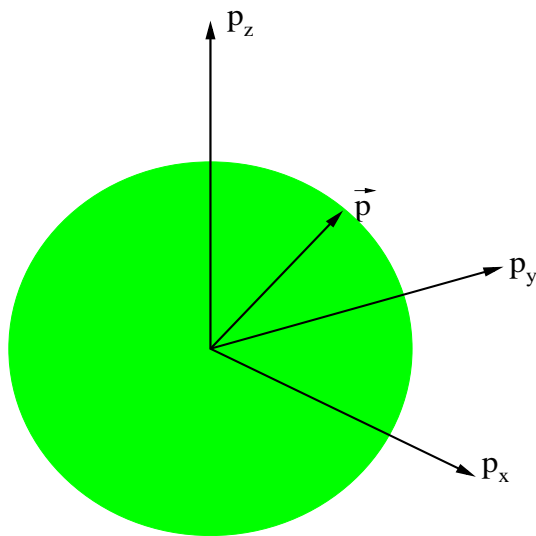


Figure 3: Momentum space: All particles with $|p|$ within $[p, p + \Delta p]$ are located on the thin shell of thickness Δp at radius $|p|$.

The Maxwell-Boltzmann distribution function for an ideal gas is

$$n(\vec{p}) = n \left(\frac{1}{2\pi mkT} \right)^{3/2} e^{-p^2/(2mkT)},$$

where n is the total number of particles per volume. This is the density $n(\vec{p})$ of particles with momentum \vec{p} . Above we needed the number density $n(p)$ of particles with an absolute value of the momentum p . Thus, we need to integrate over all possible angles of the vector \vec{p} . We can imagine that we have a momentum space, i.e. a three dimensional space with axes p_x , p_y and p_z (see figure 3). All possible momentum vectors \vec{p} are vectors pointing to a coordinate (p_x, p_y, p_z) in this momentum space. All particles which have an absolute value p of their momentum \vec{p} are located on a spherical shell at distance p from the origin in this momentum space. Thus we may imagine a particle to have a position in the six dimensional position-momentum space (\vec{x}, \vec{p}) . All particles have a position in real space (x, y, z) and a position in momentum space (p_x, p_y, p_z) . All particles with momentum between p and $p + dp$ are located on a thin shell of thickness dp at a distance p from the origin. The total volume of this shell is $4\pi p^2 dp$. Thus, to obtain the total number of particles within this momentum range, we need to multiply the

distribution with the momentum space volume $4\pi p^2 dp$,

$$n(p)dp = n \left(\frac{1}{2\pi mkT} \right)^{3/2} e^{-p^2/(2mkT)} 4\pi p^2 dp.$$

This is the distribution function for absolute momenta p . Note that whereas $n(\vec{p})$ has dimensions number density per real volume per volume in momentum space, $n(p)dp$ has dimensions number density per real volume. The latter follows from the fact that we have simply multiplied $n(\vec{p})$ with a volume in momentum space ($4\pi p^2 dp$) to obtain $n(p)dp$.

We have also, without knowing it, encountered another distribution function in this course. The Planck distribution. The Planck distribution is the number density of photons within a given frequency range

$$B(\nu) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/(kT)} - 1}.$$

When you have taken courses in quantum mechanics and thermodynamics you will deduce two more general distribution functions. When taking quantum mechanical effects into account it can be shown that the distribution function for fermions (fermions were particle with half integer quantum spin like the electron, proton or neutron) and bosons (bosons were particles with integer quantum spin like the photon) can be written generally as

$$n(E) = \frac{g(E)}{e^{(E-\mu_C)/(kT)} \pm 1},$$

where μ_C is the *chemical potential* and $g(E)$ is the *density of states* which we will come back to later. Here the minus sign is for bosons and the plus sign for fermions. In the limit of low densities it turns out (we will not show it here) that the exponential part dominates and the distribution function becomes equal for fermions and bosons. In this case the chemical potential has such a form that we get back the Maxwell-Boltzmann distribution function (compare with the above expression). Note that the expression for bosons resembles the Planck function: the Planck function can be derived from the distribution function for bosons (you will do this in later courses).

3 Degenerate gases

In the core of stars, the fermions, i.e. the electrons, is the dominating species. Therefore we will here study the distribution function for fermions and use

the + sign in the above equation. We look at an approximation of the distribution function for an electron gas at low temperature. Of course, the temperature in the core of a star is not particularly low, but we will later show that the same approximation and results are valid even for high temperatures provided we are in the high density limit. In the low temperature limit it can be shown that the chemical potential μ_C equals the so-called *Fermi energy* E_F . We will later find an expression for the Fermi energy, but first we will consider the distribution function

$$n(E) = \frac{g(E)}{e^{(E-E_F)/(kT)} + 1} \quad (4)$$

where

$$g(E) = 4\pi \left(\frac{2m_e}{h^2} \right)^{3/2} E^{1/2}, \quad (5)$$

where m_e is the electron mass. The number of electrons per volume with an energy between E and $E+dE$ in a gas with temperature T and Fermi energy E_F is now given by $n(E)dE$. The energy E of the electron may be larger or smaller than the Fermi energy E_F . We will now measure the energy of the electron in units of the Fermi energy. We define $x = E/E_F$ such that $x < 1$ when the energy is less than the Fermi energy and $x > 1$ when the energy is larger than the Fermi energy. The distribution function as a function of x , the energy in units of the Fermi energy, can thus be written

$$n(x) = \frac{g(x)}{e^{(x-1)E_F/(kT)} + 1}.$$

In the low temperature limit, $T \rightarrow 0$, the factor $E_F/(kT)$ is a very large quantity. The energy x defines whether the number in the exponential is a large positive or a large negative quantity. If $x > 1$, i.e. that the energy is larger than the Fermi energy, then the number in the exponential is a large positive number and $n(x) \rightarrow 0$. For $x < 1$, i.e. the energy is less than the Fermi energy, the number in the exponential is a large negative number. Thus the exponential goes to zero and $n(x) \rightarrow g(x)$. So for very low temperatures, there is a sharp limit at $x = 1$. For $E < E_F$ we find $n(x) = g(x)$ whereas for $E > E_F$ we find $n(x) = 0$. In figure 4 we show $n(x)/g(x)$ for lower and lower temperatures.

The physical meaning of this is that for very low temperatures, all the electrons have energies up to the Fermi energy whereas no electrons have energies larger than the Fermi energy. The Fermi energy is a low temperature

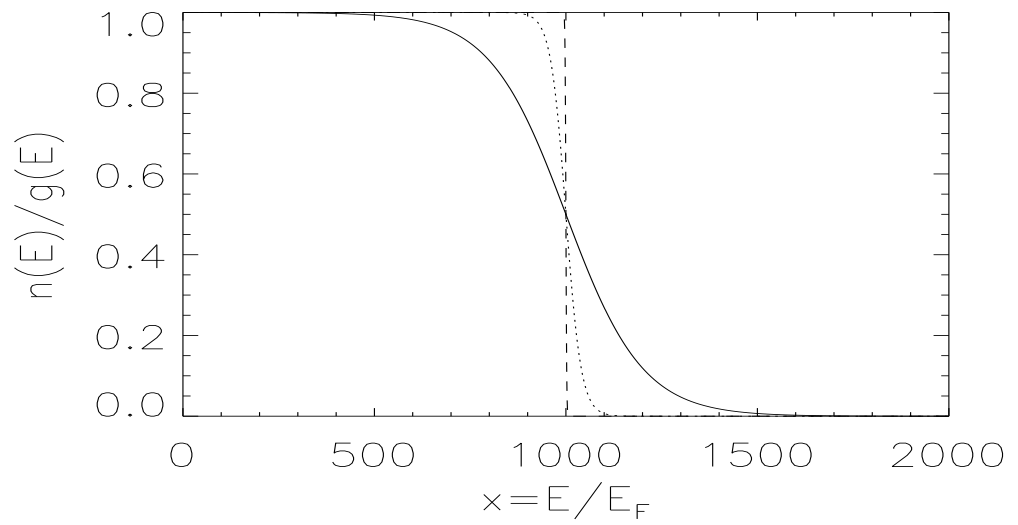


Figure 4: The number of electrons $n(E)$ divided by $g(E)$ for different energies E . The solid line is for a gas at temperature $T = 10\text{K}$, the dotted line for a gas at temperature $T = 2\text{K}$ and the dashed line for $T = 0.1\text{K}$. When the temperature approaches zero, there are less and less electrons with energy larger than the Fermi energy E_F .

energy limit for the electrons. Even if we cool an electron gas down to zero temperature, there will still be electrons having energies all the way up to the Fermi energy. But if the temperature is zero, why don't all electrons have an energy close to zero? Why don't all electrons go and occupy the lowest possible energy state allowed by quantum mechanics (in quantum mechanics, a particle cannot have zero energy)? The reason for this is hidden in quantum physics: at low temperatures the gas of electrons start to behave like a quantum gas, a gas where quantum mechanical effects are important. The quantum mechanical effect which we see on play here is the Pauli exclusion principle: Two fermions cannot occupy the same energy state. To understand this principle we need to dig even deeper into the quantum theory. According to quantum mechanics momentum is quantized. This means that a particle cannot have an arbitrary momentum. The momentum in any direction can be written as

$$p_x = N_x \cdot p_0,$$

where N_x is an integer quantum number and p_0 is the lowest possible momentum. Thus, an electron can only have x-momenta $p_0, 2p_0, 3p_0$ etc. No values in between are allowed. So the total momentum of an electron (or any particle) can be written

$$p^2 = p_x^2 + p_y^2 + p_z^2 = p_0^2(N_x^2 + N_y^2 + N_z^2) \equiv p_0^2 N^2,$$

where (N_x, N_y, N_z) are the three quantum numbers defining the state of the electron. According to the Pauli exclusion principle only one electron can occupy the quantum state (N_x, N_y, N_z) . No other electrons can have exactly the same combination of quantum numbers. We go back to the above image of a momentum space where a particle has a position (p_x, p_y, p_z) in a three dimensional momentum space in addition to a position in normal space. We can now write this position in terms of quantum numbers as $(p_x, p_y, p_z) = p_0(N_x, N_y, N_z)$. Since only one electron can have a given momentum $p_0(N_x, N_y, N_z)$, one could imagine the momentum space filled with boxes of volume $p_0 \times p_0 \times p_0$. Only one electron fits into each box. We remember that all electrons with momentum lower than a given momentum p is within a sphere with radius p in this momentum space. All electrons with a higher momentum p are outside of this sphere. But inside the sphere of radius p , there is only room for $4/3\pi N^3$ boxes of size p_0^3 (total volume of the momentum space sphere $(4/3)\pi p^3 = (4/3)\pi p_0^3 N^3$ divided by volume of box p_0^3). If all these boxes are filled, no more electron may settle on a position

inside this sphere, it has to remain outside of the sphere. When you lower the temperature of an electron gas, the electrons loose momentum and start to occupy the lowest possible momentum states, i.e. they all start to move towards the origin $(0, 0, 0)$ in momentum space. But when all start to move towards the origin in momentum space (see again figure 3), all the boxes around the origin are soon occupied, so the electrons need to remain with higher momenta at larger distances p from the origin. But if they need to remain with larger momenta, this means that they also have larger energy: The same argument therefore applies to energy. The energy states of the electrons are quantized so not all electrons may occupy the lowest energy state. For this reason we see that the distribution function for electrons at low temperatures is a step function: All electrons try to occupy the lowest possible energy state. The lowest energy states are filled up to the Fermi energy. If we call p_F the Fermi momentum, the momentum corresponding to the Fermi energy we can imagine that all electrons start to gather around the origin in momentum space out to the radius p_F . All electrons are packed together inside a sphere of radius p_F in momentum space. When you add more electrons to the gas, i.e. the density of electrons increases, the sphere in momentum space inside which all the electrons are packed also needs to expand and the Fermi momentum p_F increases. Thus the Fermi momentum and the Fermi energy are functions of the electron density n_e .

Having learned that for very low temperatures, the electrons are packed together in momentum space in a sphere of radius p_F we can find the total number density (per real space volume) n_e of electrons in the gas by summing up all the boxes of size p_0^3 inside this sphere. We know that all these boxes are occupied by one electron and that no electrons are outside this sphere. First we need to know the fermion distribution function $n(p)$ in terms of momentum rather than in terms of energy which we used above. The fermion distribution function for momentum can be written

$$n(\vec{p}) = \frac{1}{e^{(p^2 - p_F^2)/(2mkT)} + 1} \frac{2}{h^3}.$$

This is the number density per volume in real space per volume in momentum space. Considering again the low temperature case, we see, using the same arguments as before, that $n(\vec{p}) \rightarrow 0$ for $p > p_F$ and $n(\vec{p}) \rightarrow 2/h^3$ for $p < p_F$. Thus $n(\vec{p})$ is a constant for $p < p_F$ and zero for $p > p_F$. In order to obtain the number density of electrons per real space volume we need to integrate

this expression over the momentum space volume. So

$$n_e = \int_0^\infty n(\vec{p}) 4\pi p^2 dp = \int_0^{p_F} \frac{2}{h^3} 4\pi p^2 dp = \frac{8\pi}{3h^3} p_F^3$$

where we integrate over the sphere in momentum space in shells of thickness dp out to the Fermi momentum p_F where $n(\vec{p})$ is a constant ($2/h^3$) for $p < p_F$ and is zero for $p > p_F$. We use this result to obtain an expression for the Fermi momentum

$$p_F = \left(\frac{3h^3 n_e}{8\pi} \right)^{1/3}. \quad (6)$$

Using the non-relativistic expression for energy we can now find the Fermi energy expressed in terms of the electron number density n_e

$$E_F = \frac{p_F^2}{2m_e} = \frac{h^2}{8m_e} \left(\frac{3n_e}{\pi} \right)^{2/3}. \quad (7)$$

As we anticipated, the Fermi energy depends on the density of electrons. The higher the density, the larger the Fermi energy and the Fermi momentum in order to have space for all the electrons within the sphere of radius p_F . A gas where all particles are packed within this sphere so that the particles are fighting for a box in momentum space among the lowest energy states is called a *degenerate gas*. A partially degenerate gas is a gas where there are still a few vacant boxes among the lowest energy states such that some particles have energies larger than the Fermi energy. We now need to find a criterion for when a gas is degenerate.

When the temperature of a gas is high and the density low, the distribution function is the Maxwell-Boltzmann distribution function. We have previously learned that for a gas following the Maxwell-Boltzmann distribution function, the mean energy per particle is $\langle E \rangle = (3/2)kT$. The gas starts to become degenerate when most of the particles have energies below the Fermi energy. The gas therefore starts to be degenerate when the mean energy of the particles go below the Fermi energy. For an electron gas we thus have the criterion

$$\frac{3}{2}kT < E_F = \frac{h^2}{8m_e} \left(\frac{3n_e}{\pi} \right)^{2/3},$$

or

$$\frac{T}{n_e^{2/3}} < \frac{h^2}{12m_e k} \left(\frac{3}{\pi} \right)^{2/3}. \quad (8)$$

As discussed above, this criterion is satisfied for very low temperatures, but it is also satisfied for very high densities. In the exercises you will estimate what kind of densities are needed in the stellar cores for the core to be degenerate.

4 The pressure of a degenerate electron gas

When the density of electrons in the stellar core becomes high enough, most electrons have energies below the Fermi energy and the above criterion for degeneracy is satisfied. The core is electron degenerate. Now we will study the properties of a degenerate gas. The equation of state, the equation for the pressure as a function of density and temperature, is one of the most important properties describing how a gas behaves.

In order to find the pressure, we need to evaluate the pressure integral (equation 3) for the degenerate gas. First we need the density $n(p)dp$ of electrons per volume with momentum p in the interval $[p, p + dp]$. By now we have learned that $n(\vec{p})4\pi p^2 dp = n(p)dp$ such that for $p < p_F$ we have $n(p)dp = (2/h^3)4\pi p^2 dp$ and for $p > p_F$ we have $n(p) = 0$.

$$P = \frac{1}{3} \int_0^\infty p v n(p) dp = \frac{1}{3} \int_0^{p_F} \frac{p^2}{m_e} \frac{2}{h^3} 4\pi p^2 dp = \frac{8\pi}{3m_e h^3} \frac{1}{5} p_F^5.$$

Inserting the expression for the Fermi momentum (equation 6), we find

$$P = \left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{20m_e} n_e^{5/3} \quad (9)$$

We see that the pressure of a degenerate gas does not depend on the temperature. If the temperature increases or decreases, the pressure does not change! This is very different from a normal gas. It means that the degenerate stellar core will not expand or contract as the temperature changes. The only exception being when the temperature increases so much that the condition (8) for degeneracy is no longer valid and the degeneracy is broken. In this case, the electrons have gained so much energy that they are not packed in the sphere of the lowest momentum states in momentum space. The gas is no longer degenerate and a normal equation of state which depends on the temperature needs to be used.

We have deduced the pressure of a degenerate gas using the non-relativistic expressions for energy. The temperature in the stellar cores are often so high

that the velocities of the particles are relativistic. Repeating the above deductions using the relativistic expression, we would obtain

$$P = \frac{hc}{8} \left(\frac{3}{\pi} \right)^{1/3} n_e^{4/3}. \quad (10)$$

5 Summary

We have seen that if we compress a gas of fermions sufficiently, so that the degeneracy condition (equation 8) is fulfilled, the fermions are packed together inside a sphere of radius p_F in momentum space. All the lowest energy states of the fermions are occupied up to the Fermi energy E_F . This typically happens when the temperature is very low so that the fermions fall down to the lowest possible energy states in momentum space. It might also happen for high temperatures if the density is high enough: In this case there are so many fermions present within a volume so all fermion states up to E_F are occupied even if the temperature is not particularly low.

A degenerate fermion gas has a degeneracy pressure which is independent of the temperature of the gas given by equation (9) for a non-relativistic gas (the particles have non-relativistic velocities) and by equation (10) for a relativistic gas. This pressure originates from the resistance against being squeezed further together in real and momentum space and only depends on the density of the gas. We obtained the expression for the pressure by inserting the distribution function for a degenerate gas in the pressure integral (equation 3). The distribution function for a degenerate gas took on a particular form: It is a step function being constant for energies below the Fermi energy and zero above. This was simply a consequence of the Pauli exclusion principle, two energy states cannot be occupied at the same time. When the quantum states of lowest energy are occupied, the fermions need to occupy states of higher energy. For a completely degenerate gas, the Fermi energy E_F gives within which energy there is room for all fermions at a given density.

If the temperature increases sufficiently, the fermions gain enough energy to occupy states well outside the sphere of radius p_F in momentum space. Then there will be vacant low energy states, the condition of degeneracy is no longer fulfilled and the gas has become non-degenerate following a normal temperature-dependent equation of state.

6 Problems

Problem 1 In the text we used the pressure integral to find the pressure of a degenerate electron gas. Study the derivation carefully and make sure you understand every step before embarking on this exercise.

Now we want to find the pressure in a gas which follows a Maxwell-Boltzmann distribution function. Find the expression for the Maxwell-Boltzmann distribution function and use this in the pressure integral. Assume non-relativistic velocities. Remember also that the distribution function $n(p)$ used in the pressure integral needs to be normalized such that

$$\int_0^\infty n(p)dp = n,$$

where n is the number density of particles per real space volume. You now have all the information you need to find the pressure of a gas following Maxwell-Boltzmann statistics so your task is simply: find P as a function of n and T . These integrals might be useful:

$$\begin{aligned}\int_0^\infty x^{3/2}e^{-x}dx &= \frac{3\sqrt{\pi}}{4} \\ \int_0^\infty x^{1/2}e^{-x}dx &= \frac{\sqrt{\pi}}{2}\end{aligned}$$

The answer you find for P should be familiar to you.

Problem 2 In the text we found the condition for a gas to be degenerate in terms of the temperature T of the gas and the number density n_e of electrons. We will now try to rewrite this expression into a condition on the mass density ρ of the gas.

1. Assume that the gas is neutral, i.e. that there is an equal number of protons and electrons. Show that this gives

$$n_e = \frac{Z\rho}{Am_H},$$

where Z is the average number of protons per nucleus, A is the average number of nucleons per nucleus, m_H is the hydrogen mass and ρ is the total mass density.

2. Find the expression for the condition for degeneracy in terms of the total mass density ρ instead of n_e .
3. Find the minimum density a gas with temperature $T = 10^9\text{K}$ must have in order to be degenerate. A typical atom in the gas has the same number of protons and neutrons.
4. If you compress the Sun into a sphere with radius R and uniform density until it becomes degenerate, what would be the radius R of the degenerate compressed Sun? This is basically what will happen at the end of the Sun's life time. Gravitation will compress it until it becomes a degenerate white dwarf star. A white dwarf star typically has a radius similar to the radius of the Earth. Does this fit well with your result?
5. What about Earth? To which radius would you need to press the Earth in order for it to become degenerate?

Problem 3 The number density per real space volume per momentum space volume of particles with momentum \vec{p} is given by $n(\vec{p})$ found in the text. In order to find the number density per real space volume of particles with absolute momentum p we multiplied $n(\vec{p})$ with an infinitely small volume element $4\pi p^2 dp$ and obtained $n(p)dp$. Go back to the text and make sure that you understand this transition.

1. Now we will try to find the number density per real space volume of particles with energy E using the non-relativistic formula for energy $E = p^2/2m$. Start with $n(p)dp$, make the substitution and show that you arrive at equation 4 with $g(E)$ looking like equation in 5.
2. In the exercises in the previous lecture, we found that the mean kinetic energy of a particle in an ideal gas is $(3/2)kT$. Now we will try to find the mean kinetic energy in a degenerate gas. First of all, repeat what you did in the exercise in the previous lecture. Now you will repeat the same procedure, but use $n(E)$ and E directly,

$$\langle E \rangle = \int_0^\infty P(E) E dE.$$

You will need to find out how $P(E)$ looks like. The answer is

$$\langle E \rangle = \frac{3}{5} E_F.$$