The following people have participated in creating these solutions: Nicolaas E. Groeneboom, Magnus Pedersen Lohne, Karl R. Leikanger
NOTE: There might be errors in the solution. If you find something which doens't look right, please let me know

Partial solutions to problems: part 1A

Problem 1A.5

1. Do the next two exercises first. When you understand how they are done, this one is easy using the same techniques. The answer is

$$\langle v \rangle = \sqrt{\frac{8kT}{\pi m}}$$

2. We have

$$P = \frac{1}{3}\int_0^\infty pvn(p)dp$$

Starting from eq.12 in lecture notes part 1A, we have

$$P = \frac{1}{3}\int_0^\infty \frac{p^2}{m}n(p)dp = \frac{1}{3m}\int_0^\infty n\Big(\frac{1}{2\pi mkT}\Big)^{3/2} 4\pi e^{-p^2/(2mkT)}p^4 dp$$

summarizing terms:

$$P = \frac{4\pi}{3m}\Big(\frac{1}{2\pi mkT}\Big)^{3/2}\int_0^\infty ne^{-p^2/(2mkT)}p^4 dp$$

Perform the substitution $x = \frac{p^2}{2mkT}$ such that

$$p^2 = 2mkTx$$

and hence

$$dp = \frac{1}{2}\sqrt{\frac{2mkT}{x}}dx$$

Substituting:

$$P = \frac{4\pi n}{3m}\Big(\frac{1}{2\pi mkT}\Big)^{3/2}\int_0^\infty e^{-x}(2mkTx)^2\frac{1}{2}\sqrt{\frac{2mkT}{x}}dx$$

Summarizing again:

$$P = \frac{2\pi n}{3m}\Big(\frac{1}{2\pi mkT}\Big)^{3/2}(2mkT)^{5/2}\int_0^\infty e^{-x}x^{3/2}dx$$

where we have the integral

$$\Gamma(n) = n\Gamma(n-1) = \int_0^\infty e^{-x}x^{n-1}dx$$

giving the **Gamma-function**. For $n \in \mathbb{N}$, we have that $\Gamma(n+1) = n!$ and $\Gamma(1/2) = \sqrt{\pi}$ such that $\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}$. This function will become *very important* when working with statistical physics and quantum mechanics, so it's in general a good idea to get familiarized and friendly with it as soon as possible. The Γ-function doesn't bite.. too much. Using that $\Gamma(5/2) = \frac{3}{2}\Gamma(\frac{3}{2}) = \frac{3}{4}\sqrt{\pi}$, then

$$P = \frac{\pi^{3/2} n}{2m}\Big(\frac{1}{2\pi mkT}\Big)^{3/2}(2mkT)^{5/2} = nkT$$

3. We begin by determining the average energy of the gas:

$$\langle E\rangle = \langle\frac{1}{2}mv^2\rangle = \frac{1}{2}m\langle v^2\rangle = \int_0^\infty P(v)\frac{1}{2}mv^2 dv$$

Inserting all values yields

$$\langle E\rangle = \frac{m}{2}\Big(\frac{m}{2\pi kT}\Big)^{3/2} 4\pi \int_0^\infty e^{-\frac{1}{2}\frac{mv^2}{kT}} v^4 dv$$

Perform the substitution $x = \frac{1}{2}\frac{mv^2}{kT}$ such that

$$v^2 = \frac{2xkT}{m}$$

and hence

$$dv = \frac{kT}{vm}dx = \frac{kT\sqrt{m}}{m\sqrt{2xkT}}dx = \sqrt{\frac{kT}{2mx}}dx$$

Inserting

$$\langle E\rangle = \frac{m}{2}\Big(\frac{m}{2\pi kT}\Big)^{3/2} 4\pi \int_0^\infty e^{-x}\Big(\frac{2xkT}{m}\Big)^2\sqrt{\frac{kT}{2mx}}dx$$

or

$$\langle E\rangle = \frac{m}{2}\Big(\frac{m}{2\pi kT}\Big)^{3/2} 4\pi\Big(\frac{2kT}{m}\Big)^2\sqrt{\frac{kT}{2m}}\Gamma(\frac{5}{2})$$

$$\langle E\rangle = \frac{m}{2}\Big(\frac{m}{2\pi kT}\Big)^{3/2} 4\pi\Big(\frac{2kT}{m}\Big)^2\sqrt{\frac{kT}{2m}}\frac{3}{4}\sqrt{\pi}$$

Summarizing:

$$\langle E\rangle = \frac{3}{2}m\Big(\frac{m}{2\pi kT}\Big)^{3/2}\Big(\frac{2\pi kT}{m}\Big)^{3/2}\Big(\frac{kT}{m}\Big)$$

or (finally)

$$\langle E\rangle = \frac{3}{2}kT$$