

Lecture notes 2: Coordinate systems

Any plane passing through the center of a sphere cuts the surface in a circle which is called a *great circle*. Any other plane that cuts the sphere, but that does not pass through the center is a *small circle*. When two great circles intersect at a point they are said to include a *spherical angle* which is defined between the tangents of the great circles at the point of their intersection. A spherical angle is only defined with respect to intersecting great circles.

Given any three points on the surface of a sphere, the sphere can be bisected so that all three points lie in the same hemisphere. Joining the points by great circle arcs all in this hemisphere defines a *spherical triangle*. The length of a great circle arc is defined as the radius times the angle A formed between the two endpoints of the arc and the sphere's center measured in radians: $R \times A$.

There are several methods of specifying a given position on the celestial sphere, depending on which principal great circles are chosen as reference.

Altitude – Azimuth

With reference to figure 1 let O , the observer on the surface of the earth (supposed spherical), be the center of the celestial sphere.

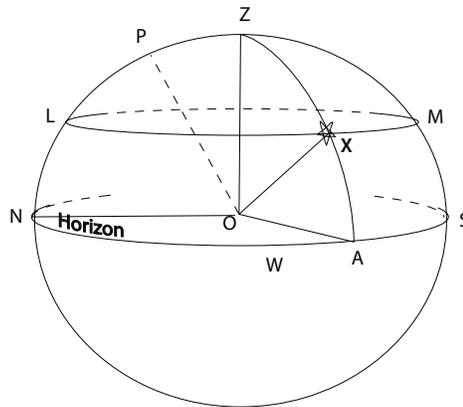


Figure 1: The altitude – azimuth coordinate system.

Let Z be the zenith, vertically overhead be defined by the direction of gravity. OZ is thus the continuation of the straight line joining the earth's center to O . The plane through O at right angles to OZ is the plane of the horizon, cutting the celestial sphere in the great circle NAS , called the *celestial horizon*.

Let X be the position of a star on the celestial sphere. Any great circle drawn through Z is called a *vertical circle*; in particular, the vertical circle through X is ZXA . In the plane of ZXA , the angle AOX (or the great circle arc AX is called the *altitude* denoted by a . Since OZ is perpendicular to the plane of the

horizon, the great circle arc ZA is 90° ; hence $ZX = 90^\circ - a$. ZX is called the *zenith distance* of the star X . Draw LXM as a small circle through X parallel to the horizon, it is called the *parallell of altitude*.

To define a stars position completely on the celestial sphere the particular vertical circle on which it lies must also be specified. Let OP be parallel to the axis about which the earth spins. On the northern hemisphere the position P is called the *north celestial pole*. Due to this rotation the celestial sphere appears to rotate and the stars to continuously change altitude and direction. In the northern hemisphere Polaris lies almost directly on OP and changes direction very little. Define the vertical circle through P that is ZPN as the principal vertical circle and the point N as the *north point of the horizon*.

The position of a star X on the celestial sphere at a given moment by reference to the horizon and the principal vertical circle ZPN . If the star is in the western part of the celestial sphere the spherical angle PZX or the great circle arc NA is called the azimuth (W).

Note that the angle POZ (or great circle arc PZ) is equivalent to the angle between the radius of the earth which passes through the observer's position and the earth's axis is equal to the colatitude of the observer or

$$PZ = 90^\circ - \phi$$

where ϕ is the observers latitude. Hence the altitude of the pole is equal to the observers latitude.

Some spherical trigonometry

Both great circle angles and arcs are measured in radians (or degrees). By definition all angles and sides in a spherical triangle are less than π radians (180°). A spherical triangle is defined when we know three of its six sides or angles. The sum of the angles in a spherical triangle is greater than 180° , the difference between the sum of the angles and 180° is called the *spherical excess*.

Consider a spherical triangle with corners A, B, C on a unit sphere. The sides a, b, c lie directly opposite these corners. Insert a right angled coordinate system x, y, z with origin at the sphere's center, let the z axis go through the point A , and the $x - z$ plane through the side c . The coordinates of the corner C are then

$$\begin{aligned} z &= \cos b \\ x &= \sin b \cos A \\ y &= \sin b \sin A. \end{aligned} \tag{1}$$

Now rotate the coordinate system around the y -axis until the z -axis goes through B . Call the new coordinate system x', y', z' . The corner C 's coordinates in the new system are then - since we see that the side a is equivalent to b in the old system and the angle $\pi - B$ is equivalent to A :

$$z' = \cos a$$

$$\begin{aligned} x' &= -\sin a \cos B \\ y' &= \sin a \sin B. \end{aligned} \tag{2}$$

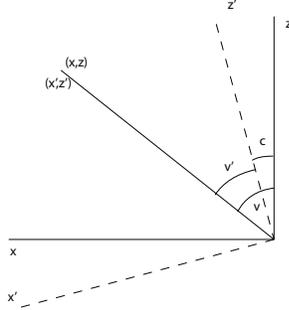


Figure 2: Two axis systems used to derive the basic formula of spherical trigonometry.

To find the relation between the marked and unmarked coordinates, put a plane through the $x - z$ axes. On rotation the y -coordinate is unchanged. With reference to figure 2 we see the following relations:

$$\begin{aligned} v' &= v - c & z &= \cos v & x &= \sin v \\ z' &= \cos v' & x' &= \sin v' \end{aligned}$$

Now express z' and x' with the help of v and c

$$\begin{aligned} z' &= \cos v \cos c + \sin v \sin c \\ x' &= \sin v \cos c - \sin c \cos v, \end{aligned}$$

setting in for z and x and remembering that y is unchanged gives

$$\begin{aligned} z' &= z \cos c + x \sin c \\ x' &= x \cos c - z \sin c \\ y' &= y. \end{aligned} \tag{3}$$

Setting in equations 2 and 3 in equation 4 gives the basic formulae fo spherical trigonometry:

$$\cos a = \cos b \cos c + \sin b \sin c \cos A \tag{4}$$

$$\sin a \cos B = \sin c \cos b - \cos c \sin b \cos A \tag{5}$$

$$\sin a \sin B = \sin b \sin A \tag{6}$$

The first of these is the *cosine formula*, the second the *sine-cosine* formula, and the third the *sine* formula.

the star moves along the small circle LFM crossing the horizon at F , when it is said to *set* with an altitude of 0° . Eventually it will pass through its position of maximum depression at M before continuing on to G where the altitude again is 0° and is said to *rise*. At any moment the star's position on the parallel of declination is specified by the angle P between the observer's meridian and the meridian (PXQ) through the star at this time; this angle is RPX or arc RD on the equator. This angle, denoted by H , is called the *hour angle* and is measured from the observer's meridian *westwards* from 0° (at L) to 360° (when the star returns to the observer's meridian), or more usually from 0^h to 24^h .

The standard geocentric celestial sphere

For stars, which are very far away, the position of the observer on the surface of the earth is irrelevant to the definition of the stars position. But when we are considering a relatively nearby object such as the moon, the sun, or a planet the definition of the north polar distance (and therefore of declination) previously given *is* dependent on position of the observer on the earth.

Accordingly, the center of the standard celestial sphere is taken to be at C the earth's center.

Right ascension and declination

In the hour angle and declination method of specifying a star's position on the celestial sphere only one coordinate, the declination, remains constant as the star travels across the sky.

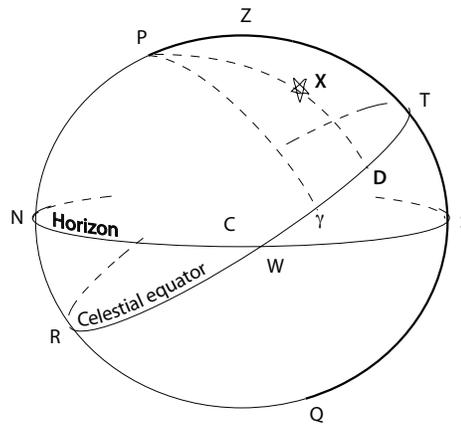


Figure 4: Definition of the right ascension – declination coordinate system.

Let us now pick a point Υ on the celestial equator and let the meridian through the star X cut the celestial equator at D . As the stars pass across the sky the declination DX remains constant and that the relative configuration of

the stars also remain constant. It follows that ΥD is constant. We regard Υ as a reference point on the celestial equator; we can then clearly specify the position of the star X by means of the great circle arc ΥD and the declination DX . The reference point chosen is called the *vernal equinox* or the *first point of aries*. The arc ΥD is called the *right ascension* (R.A.) of the star X . It is denoted by α and is measured *eastward* from Υ from 0^h to 24^h , opposite to the direction the hour angle H is measured.

Note that $R\Upsilon = RD + \Upsilon D$. The hour angle of Υ is called the *sidereal time* (S.T.). We have accordingly

$$\text{S.T.} = H + \alpha$$

When Υ is on the observer's meridian, the hour angle of Υ is 0^h , *i.e.* the sidereal time is 0^h . When Υ is next on the observer's meridian an interval of 24^h of sidereal time has elapsed. This interval is the same as is required for the complete revolution of the earth about its axis and is called a *sidereal day*.

The earth's orbit

According to Kepler's first law of planetary motion, the earth is a planet revolving around the sun in an elliptical path or orbit, and the sun is situated in a focus of the ellipse. Since our observations are made from earth, then relative to the earth the sun appears to describe an elliptical path around the earth. In the course of a year the sun makes a complete circuit of the heavens against the background of the stars. The plane of the orbit is called the *plane of the ecliptic*, and the great circle in which this plane intersects the celestial sphere whose center is the earth's center C , is called the *ecliptic*. With reference to the stars, the plane of the ecliptic will be a particular great circle which is found by observations to be inclined at an angle of about $23\frac{1}{2}^\circ$ to the celestial equator.

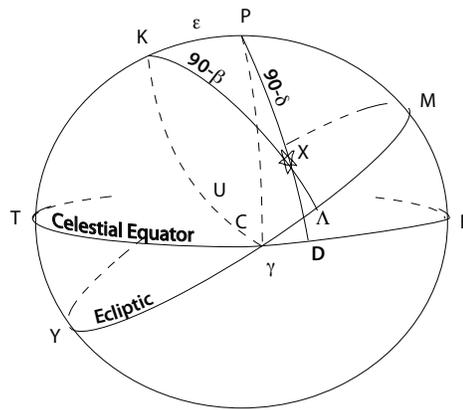


Figure 5: Celestial longitude and latitude.

In figure 5, $Y\Upsilon M$ represents the ecliptic and its inclination to the celestial equator is $M\Upsilon R$, which is known as the *obliquity of the ecliptic*. Relative to the earth, the sun appears to move on the celestial sphere along the ecliptic in the direction $Y\Upsilon M$ and twice yearly, its position coincides with the intersections of the ecliptic with the celestial equator. The position Υ , at which the sun's declination changes from south to north, is the *vernal equinox*. It is in this way that the reference point Υ is obtained, from which the right ascension of stars is measured. From the diagram it is seen that the right ascension and declination of the sun are both changing continually. When the sun is at Υ its right ascension and declination are both zero (this occurs roughly March 21); at M the right ascension is 6^h and declination about $23\frac{1}{2}^\circ$ N (roughly June 21, summer solstice); at U the right ascension is 12^h and declination 0° (September 21, autumnal equinox), and at Y the right ascension is 18^h and the declination about $23\frac{1}{2}^\circ$ (December 21, winter solstice).

Celestial latitude and longitude

The position of a heavenly body can also be referred to the ecliptic as fundamental great circle and the vernal equinox Υ as principal reference point. E.g. in figure 5 K is the north pole of the ecliptic and KXA is a great circle arc passing through X and meeting the ecliptic in Λ . The arc $\Upsilon\Lambda$, measured from Υ along the ecliptic in the direction of the sun's annual motion (eastwards), is called the *longitude* of X and is measured from 0° to 360° round the ecliptic. The arc ΛX is the *latitude*, north is considered positive and south negative.

Thus, if one know a star's right ascension and declination it is possible to obtain its latitude (β) and longitude (λ) from the triangle KPX ; and vice versa.

Sidereal time I

Sidereal time at Greenwich is given such that

$$\text{S.T. at Greenwich} = \text{S.T. at } l \pm \text{long. of } l$$

where l is the longitude of the observer with the + sign given when l is west of Greenwich and the - is given when l is east of Greenwich. The sidereal time at l is called the *local sidereal time* (L.S.T.).

Mean solar time

When the sun is on the meridian of a given place, it is apparent *noon* there; when the sun is next on the meridian, an *apparent solar day* has elapsed. An apparent solar day is not constant — due to the fact that the sun's apparent orbit around the earth is not a circle but rather an ellipse. In addition the sun moves along the ecliptic and not the celestial equator so its right ascension does not increase uniformly. The average apparent solar day through the year is called a *mean solar day*. The means sun is assumed to move in the *celestial equator* at a uniform rate around the earth. The rate is such that the mean

sun completes its orbit in the same amount of time as the real sun needs to complete an orbit around the ecliptic. Assume that the right ascension of the sun is known, then

$$\text{Sid.time} = \text{H.A.M.S.} + \text{R.A.M.S}$$

The mean sun is related to the true sun by certain principles that will be discussed later; for now let us define the difference as the *equation of time* E such that

$$E = \text{R.A.M.S} - \text{R.A.}\odot$$

E can be positive or negative and varies in a complicated manner.

When the mean sun is on the meridian of Greenwich, it is *Greenwich mean noon*. The hour angle of the mean sun at Greenwich is denoted G.M.A.T. (*Greenwich mean astronomical time*). Mean time reckoned from midnight at Greenwich is called *Greenwich Mean Time* (G.M.T.), now designated *Universal Time* (U.T.). Thus

$$\text{U.T.} \equiv \text{G.M.T.} = \text{G.M.A.T.} + 12^h$$

and similarly for any place keeping the mean time appropriate to its meridian.

Sidereal time II

Sidereal time at any instant is at a given place is the hour angle of the vernal equinox. In section the ecliptic and celestial equator were considered as fixed great circles on the celestial sphere, and thus the vernal equinox as a fixed point. However, there are both the phenomena of *precession* and *nutation* so the celestial equator cannot be considered as a fixed great circle, and the position of the vernal equinox must be treated as a time varying quantity, slowly moving according to well established principles with reference to the background stars.

We will continue to assume the ecliptic as a fixed great circle. Owing to precession, the north celestial pole P describes a small circle about the pole K (see figure 5) of the ecliptic in a period of about 26 000 yr. At present P is within 1° of the star α Ursae Minoris (Polaris), but their relative positions are changing and in 12 000 yr P will be within a few degrees of Vega. It is the direction of the earth's axis that is altering continuously with reference to the background stars. Referring to figure 6 Υ is the vernal equinox for, say 1900.0 and Υ_1 the vernal equinox for 1901.0. Υ and Υ_1 are called the *mean equinoxes* at the dates in question, and the corresponding celestial equators are called the *mean equators*.

Assuming that owing to precession the north celestial pole moves uniformly along the small circle arc PP_1 and that the mean equinox moves uniformly backwards along the ecliptic from Υ to Υ_1 . It is found that the motion of Υ along the ecliptic is at the rate of 50.3 arcsec per annum.

When we define sidereal time in relation to the moving equinox, we can no longer regard the earth's rotational period to be the interval between two

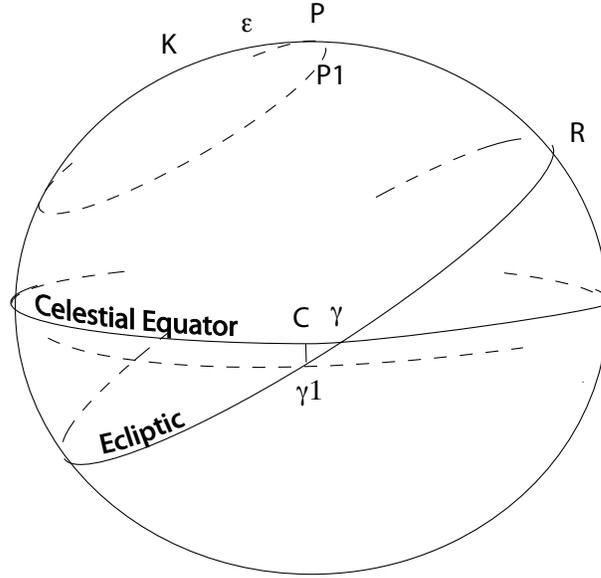


Figure 6: Definition of sidereal time and the effects of precession; motion of the vernal equinox.

successive transits of the equinox. In figure 6 let $C\Upsilon_1$ be a great circle arc drawn through Υ_1 perpendicular to the equator. Then the equinox at any given date is separating, in right ascension, from the equinox Υ for 1900.0 at the annual rate measured by ΥC , given by the small triangle formula

$$\Upsilon C = \Upsilon\Upsilon_1 \cos \varepsilon.$$

Hence, the mean equinox is separating, in right ascension, from Υ at the rate of 0.008 s per sidereal day. The direction of motion of the equinox is westward in the sky — opposite to that in which right ascension increases — and thus the interval between two successive transits of the moving equinox is 0.008 s less than the interval given by a fixed equinox. This first interval is a *sidereal day*, the second interval is the rotational period of the earth.

Owing to nutation the true equator at any instant is slightly different from the mean equator at that instant. Consequently the true equinox is displaced slightly along the ecliptic relative to the mean equinox; these displacements are periodic in nature (and due to the effect of the moon), with a period of about 18 yr. The difference in right ascension between the true equinox and the mean equinox due to this effect can amount to 1.2 s

One defines *mean sidereal time* to be associated with the moving mean equinox (only precession considered) and *apparent sidereal time* to be associated with the true equinox. The difference between these from day to day is so small in practice that generally the sidereal day is taken to mean the interval between two successive transits of the mean equinox.

Ephemeris and Universal time

There is a slight distinction between universal time, which is defined by the rotation of the earth and *ephemeris time* which is uniform and is defined by the gravitational dynamics of the solar system, independent of the earth's rotation.

When the movements of the equator and equinox due to precession are taken into account, the *fictitious mean sun* is defined to travel along the mean equator in such a way that that its mean right ascension is always equal to the sun's mean longitude. This right ascension is, therefore, independent of the rotation of the earth's, and the fictitious mean sun is a suitable reference point for the definition of ephemeris time. To facilitate this an alternate meridian is defined called the ephemeris meridian which corresponds to the sidereal direction that the Greenwich meridian would have if the earth were rotating strictly uniformly. Ephemeris time is defined as

$$\text{E.T.} = 12^h + \text{E.H.A.F.M.S.}$$

where the last term of the right hand side is the ephemeris hour angle of the fictitious mean sun.

In contrast, a slightly different reference point, the mean sun, is used to define universal time. The mean sun also moves round the mean equator but at a rate that is directly proportional at each instant to the earth's angular velocity.

Terrestrial Time and Barycentric Coordinate Time

Astronomers stuck with ephemeris time until 1979, when they defined two new time scales that used the atomic second and that took into account relativity (velocity affects time). From 1 January 1984, these scales replaced ephemeris time in national ephemerides like the Nautical Almanac.

Terrestrial Dynamical Time (TDT) views time from the earth's position and motion. It was defined as being equal to TAI (Atomic time) plus 32.184 (atomic) seconds at the instant beginning 1 January 1977.

Barycentric Dynamical Time (TDB, from the French) is time at the center of mass of the solar system. TDB has various forms depending on the theory of relativity adopted.

By International Astronomical Union (IAU) Resolution A4 in 1991, Terrestrial Dynamical Time was renamed Terrestrial Time (TT). Recommendations III and V of the same resolution created Barycentric Coordinate Time (TCB) to take the place of Barycentric Dynamical Time, except in situations where maintaining continuity in ongoing work made retaining the old scale preferable.

In 2006 (Resolution B3)¹, responding to the "multiple realizations of TDB" and other factors, the IAU defined TDB in terms of TCB. One result is that, within a few thousand years around the present, the difference between Terrestrial Time and Barycentric Dynamical Time on the surface of the Earth is less than 2 milliseconds.

¹www.iau.org/static/resolutions/IAU2006_Resol13.pdf

The sidereal year and the tropical year

The time required by the sun to make a complete circuit of the ecliptic is called a *sidereal year*.

The *tropical year* is the average interval between two consecutive passages of the sun through the vernal equinox. Thus if Υ is the position of the equinox at a given time and Υ_1 the position of the equinox one year later, the tropical year is the time taken by the sun to describe 360° less $\Upsilon_1\Upsilon$. From observations it is found to be 365.2422 days.

The relation between the sidereal year and the tropical year is then evidently

$$\text{Sid. year/trop. year} = 360^\circ / (360^\circ - 50.3'')$$

This gives a sidereal year of 365.2564 days.

During the course of a tropical year the R.A.M.S increases from 0° to 360° , that is at the rate of $360^\circ/365.2422$ or $59'8.33''$ per mean solar day. Let t_1 be the mean sidereal time when the hour angle of the mean sun at a given place is H_1 and let R_1 denote the corresponding value of the R.A.M.S.. Then

$$t_1 = H_1 + R_1$$

Let t_2 be the mean sidereal time one mean solar day later. The hour angle of the mean sun has increased by 360° and the R.A.M.S by $59'8.33''$. Hence,

$$t_2 = (H_1 + 24^h) + (R_1 + 3^m56.556^s)$$

when we convert these to time measure, so that

$$t_2 - t_1 = 24^h3^m56.556^s$$

This means that 24^h U.T. is equal to $24^h3^m56.556^s$ mean sidereal time. Note that this is equivalent to noting that during a year the earth has rotated about its axis 365.2422 times with respect with the mean sun and once more with respect to the equinox.

The Besselian year

It is the general astronomical practice to define the beginning of the tropical year (sometimes called the solar year) as the instant when the R.A. of the fictitious mean sun is exactly 18^h40^m or 280° . This instant falls near the beginning of the civil year and is usually called the *Besselian year*.

It is general practice to denote the beginning of any Besselian year by the notation *e.g.* 1975.0, 2008.0, etc.

The Julian date

In certain observations it is found convenient to express the instant of observations as so many days and fraction of a day after a definitive fundamental epoch. The epoch chosen is Greenwich mean noon of January 1, 4713 B.C., and for any given date the number of days which have elapsed since this epoch defines the *Julian Date* (J.D.) of the date in question.

Exercises

1. Given the observers latitude ϕ , the declination δ and hour angle H of the heavenly body, calculate the zenith distance z and the azimuth A .
2. Given the observer's latitude ϕ , the stars zenith distance and azimuth, calculate the star's declination and hour angle.
3. Work out a stars latitude (β) and longitude (λ) given its declination (δ), right ascension (α) and the obliquity of the ecliptic (ε).