

Project 3

AST3220, 2025

Read this before you start

This project consists of a set of individual problems spanning different topics from the course. You can just answer the questions one by one, no need to structure a cohesive report. It is important that you explain how you think, just writing down a bunch of equations with no explanations will not give you a full score.

You should write your answers using LaTeX. Posting handwritten lecture notes and solutions to problems is a privilege that belongs to the lecturer alone. Results and equations taken from the lecture notes need not be referenced, but other sources that you use must be cited. You can work together to solve the problems, but you must write your own report. As usual, you are only allowed to use real intelligence, no AI.

Miscellaneous problems

Problem 1 (10 points)

In this problem you can assume that the Universe is described by the Einstein-de Sitter model.

- a) Write down the expression for the scale factor as a function of time. Use this expression to show that the age of the Universe at redshift z is given by

$$t(z) = \frac{t_0}{(1+z)^{3/2}},$$

where t_0 is the present age of the Universe.

- b) Assume that we at the present epoch observe two objects, one with redshift $z_1 = 3$ and another with redshift $z_2 = 8$. When, in units of t_0 , was the light we observe today emitted by these objects?
- c) Determine the comoving radial coordinates of the two objects.
- d) The light we receive now from the object with $z = z_2 = 8$ was emitted at a time we can call t_e . Determine the comoving radial coordinate of the light ray heading towards us at an arbitrary later time t .
- e) Imagine that there was an observer situated at the object with $z = z_1 = 3$. What redshift did she measure for the light from the object we observe today with redshift $z_2 = 8$?

Problem 2 (10 points)

- a) Use the Friedmann equations to explain why there must be a time in the past when the scale factor vanished in models where the total density and pressure satisfy $\rho + 3p/c^2 > 0$, the density decreases faster with a than $1/a^2$, and $H_0 > 0$.
- b) We seem to be living in a universe where the expansion is presently accelerating, $\Omega_{m0} = 0.3$, and $\Omega_{\Lambda 0} = 0.7$. Was there a time in the past when $a = 0$ in this model? Why/why not?

Problem 3 (10 points)

The proper distance to the particle horizon at time t is given by

$$d_{P,PH}(t) = a(t) \int_0^t \frac{cdt'}{a(t')}.$$

- a) Show that the proper distance to the particle horizon at redshift z is

$$d_{P,PH}(z) = \frac{c}{1+z} \int_z^\infty \frac{dz'}{H(z')}.$$

- b) Find $d_{P,PH}(z)$ for respectively a matter-dominated universe with

$$H(z) = H_0 \sqrt{\Omega_{m0}} (1+z)^{3/2},$$

and for a radiation-dominated universe with

$$H(z) = H_0 \sqrt{\Omega_{r0}} (1+z)^2.$$

Show that in both cases,

$$d_{P,PH}(z) \sim \frac{c}{H(z)}.$$

A typical neutron star has a radius of around 10 kilometers, and a mass around 1.5 solar masses. Take the radius of the observable universe at any given time to be equal to the proper distance to the particle horizon at that time.

- c) At what redshift was the radius of the observable universe equal to that of a typical neutron star? What was the mass density of the universe at that redshift? What was the radiation density (use units of kilograms per cubic meter)? Assume $\Omega_{m0} = 0.3$ and $\Omega_{r0} = 10^{-4}$. How does these densities compare with the average density of a typical neutron star. Assume that the dimensionless Hubble constant is $h = 0.7$.
- d) What was the CMB temperature at that time?
- e) Show that the age of the Universe at redshift z is given by

$$t(z) = \int_z^\infty \frac{dz'}{(1+z') H(z')},$$

and that for the radiation-dominated universe

$$t(z) = \frac{1}{2H(z)}.$$

How old was the Universe at the redshift found in c)?

On inflation

The following problems are all about inflation, and we will use units where $\hbar = 1$ and $c = 1$. In addition we will also simplify the equations by introducing the so-called reduced Planck mass $M_{\text{P}} = 1/\sqrt{8\pi G}$.

Problem 4 (5 points)

Give a brief description of the horizon problem and the flatness problem.

Problem 5 (5 points)

Assume inflation is driven by a scalar field with the potential

$$V(\phi) = \lambda\phi^p,$$

where λ is a constant and $p \geq 2$. Show that the total number of e-foldings during inflation is guaranteed to be large if the slow-roll conditions are fulfilled.

Problem 6 (5 points)

- a) Is inflation possible if $V(\phi) = 0$ for all ϕ ?
- b) Is inflation possible if the dynamics of the scalar field is such that we always have $\dot{\phi}^2 = 2V(\phi)$?

Problem 7 (10 points)

We will look at inflation driven by a scalar field with the potential $V(\phi) = V_0 e^{-\lambda\phi}$, where V_0 and λ are positive constants.

- a) Write down the equations for ϕ and H in the slow-roll approximation with this potential.
- b) Solve the equations from a) and find $\phi(t)$ and $a(t)$.
- c) The full equations, without the slow-roll approximation, are

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \tag{1}$$

$$H^2 = \frac{1}{3M_{\text{P}}^2} \left[\frac{1}{2}\dot{\phi}^2 + V(\phi) \right] \tag{2}$$

and in the case of the exponential we consider in this problem it turns out that they have an exact analytical solution. The solution you found in b) suggests the ansatz

$$a(t) = Ct^\alpha$$
$$\phi(t) = \frac{2}{\lambda} \ln(Bt).$$

Determine the constants α and B that make this ansatz an exact solution, and show that you regain the slow-roll solution in the appropriate limit.

- d) What is the main problem with this potential as a model of inflation?

Problem 8 (20 points)

The slow-roll approximation turns the second-order differential equation for the scalar field into a first-order equation. This can only work if the dynamics of the field makes the precise initial conditions on the field (apart from the fact that the field must start out somewhere on the potential where the slow-roll conditions are satisfied) redundant. In this problem you are going to set up an argument for why this will usually be the case. Without loss of generality, you can assume that $\dot{\phi} > 0$ during inflation.

- a) Use eqs. (1) and (2) from problem 7 to show that

$$\dot{\phi} = -2M_{\text{P}}^2 H'(\phi),$$

where $H'(\phi) = \frac{dH}{d\phi}$.

- b) Use the result from a) to show that the first Friedmann equation can be written as

$$[H'(\phi)]^2 - \frac{3}{2M_{\text{P}}^2} H^2(\phi) = -\frac{1}{2M_{\text{P}}^4} V(\phi) \quad (3)$$

The scalar field ϕ is in itself not directly observable, the important physical quantity is the Hubble parameter. If we can show that we end up on the same curve in the ϕ - H plane regardless of the initial conditions on the scalar field, we have the result we want.

Consider a linear perturbation around a solution of eq. (3) of the form

$$H(\phi) = H_0(\phi) + \delta H(\phi) \quad (4)$$

where H_0 is the said original solution (**note: not the Hubble constant!**).

- c) Assume that H of eq. (4) is also a solution of eq. (3), and show that to first order in the perturbation we have

$$H_0' \delta H' = \frac{3}{2M_{\text{P}}^2} H_0 \delta H$$

where $'$ again denotes derivatives with respect to ϕ .

- d) Show that the equation in c) has the general solution

$$\delta H(\phi) = \delta H(\phi_i) \exp \left[\frac{3}{2M_{\text{P}}^2} \int_{\phi_i}^{\phi} \frac{H_0(\varphi)}{H_0'(\varphi)} d\varphi \right] \quad (5)$$

where ϕ_i is the initial value of the scalar field, and φ is just a dummy integration variable. Use this result to explain why the perturbation δH quickly dies out.

Cosmological parameters

Problem 9 (20 points)

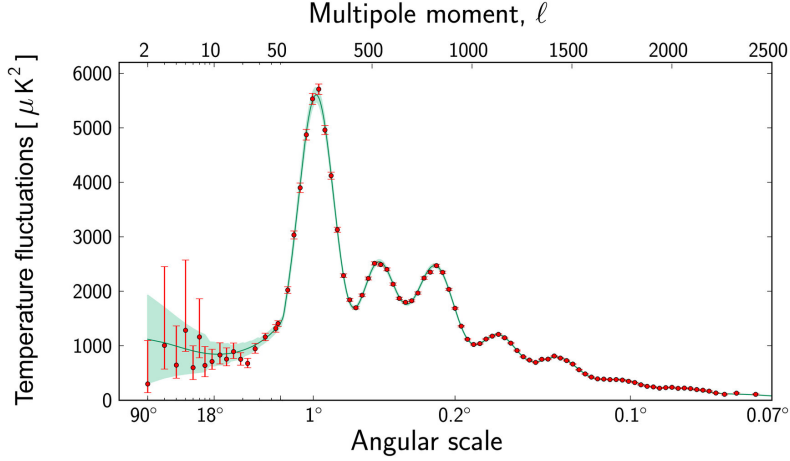


Figure 1: The best-fit Λ CDM Planck temperature power spectrum.

The figure above shows the CMB power spectrum. In the following we will consider how changes in the cosmological parameters affect the CMB spectrum. You just need to explain the main effect associated to the specific cosmological parameter assuming the other ones are kept constant.

- What would happen to the CMB power spectrum if one increases Ω_b ? Draw a cartoon to illustrate this. Explain this effect physically.
- What would happen to the CMB power spectrum if one increases Ω_k ? Draw a cartoon to illustrate this. Explain this effect physically.
- What would happen to the power spectrum if one increases Ω_Λ . Draw a cartoon to illustrate this. Explain this effect physically.
- What would happen to the CMB spectrum if you *increase* the CMB temperature, T_0 ? Draw a cartoon to illustrate this. Explain this effect physically.
- What would happen to the CMB power spectrum if one increases A_s (primordial power spectrum amplitude) *and* the spectral tilt n_s ? Draw a cartoon to illustrate this. Explain this effect physically.