2 Saha-Boltzmann calibration of the Harvard sequence ("Cecilia Payne")

Figure 4: Cecilia Payne (1900 – 1979) was educated at Cambridge by Milne and Eddington. She went to the US in 1923 and spent the rest of her career at Harvard (Boston). Her 1925 thesis was the first one in astronomy at Harvard University and remains highly readable as a wide review of stellar spectroscopy at the time. The main conclusion was that stellar composition does not change much from star to star. Russell had already suggested so a decade earlier, but her thesis under Russells’ guidance, published as the first Harvard Observatory Monograph, brought the point home. Copied from Hearnshaw (1986).
The Boltzmann and Saha laws

Boltzmann distribution:

\[
\frac{n_{r,s}}{N_r} = \frac{g_{r,s}}{U_r} e^{-\chi_{r,s}/kT}
\]  

(1)

with partition function \( U_r \):

\[
U_r \equiv \sum_s g_{r,s} e^{-\chi_{r,s}/kT}
\]  

(2)

and \( n_{r,s} \) the particle number density in level \( s \) of ionization stage \( r \). \( \chi_{r,s} \) is the excitation energy of that level, measured from the ground state (\( r, s = 1 \)).

Saha distribution:

\[
\frac{N_{r+1}}{N_r} = \frac{1}{N_e} \frac{2U_{r+1}}{U_r} \left( \frac{2\pi m_e kT}{\hbar^2} \right)^{3/2} e^{-\chi_r/kT}
\]  

(3)

with \( \chi_r \) the threshold ionization energy to ionize from stage \( r \) to \( r + 1 \).
Lyman, Balmer, Paschen, Brackett series: transitions between energy levels in **neutral** Hydrogen
Payne’s basic assumption was that the strength of the absorption lines in stellar spectra scales with the population density of the lower level of the corresponding transition. With this assumption, we can give rough estimates of the strength ratios of the $\alpha$ lines in the H I Lyman, Balmer, Paschen and Brackett series. Note that for Hydrogen, the statistical weight goes as $2s^2$ (see Sect. 2.7).

<table>
<thead>
<tr>
<th>line</th>
<th>$s$</th>
<th>$\chi_{1,s}$ [eV]</th>
<th>$g_{1,s}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lyman $\alpha$</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>Balmer $\alpha$</td>
<td>2</td>
<td>10.20</td>
<td>8</td>
</tr>
<tr>
<td>Paschen $\alpha$</td>
<td>3</td>
<td>12.09</td>
<td>18</td>
</tr>
<tr>
<td>Brackett $\alpha$</td>
<td>4</td>
<td>12.75</td>
<td>32</td>
</tr>
</tbody>
</table>

Strength ratio Lyman $\alpha$ / Balmer $\alpha$:

$$\frac{n_{1,1}}{n_{1,2}} = \frac{g_{1,1} \exp^{-\chi_{1,1}/kT}}{g_{1,2} \exp^{-\chi_{1,2}/kT}} = 2 \cdot 10^8$$

for a solar temperature of 5777 K.

Strength ratio Balmer $\alpha$ / Paschen $\alpha$: 20

Strength ratio Balmer $\alpha$ / Brackett $\alpha$: 42

lower level populations (Boltzmann)
Schadeenium: simple atom so that we can evaluate Saha-Boltzmann statistics without bothering about complex atomic data.

Figure 8: Energy level diagram for Schadee’s element E, showing the neutral stage (left hand column, \( r = 1 \)) and the first three ionization stages (\( r = 2 - 4 \)). The level energies increase in 1 eV steps. The columns may be thought of as being stacked on top of each other since each ion requires the previous stage to be ionized. The level counter \( s \) starts at 1 within each stage (but IDL starts at 0, as do the level energies). In astronomical convention the spectra of neutral schadeenium \( E \), ionized schadeenium \( E^+ \) and doubly ionized schadeenium \( E^{2+} \) are called \( EI \), \( EII \), and \( EIII \), respectively.

- ionization energies \( \chi_1 = 7 \text{ eV} \) for neutral \( E \), \( \chi_2 = 16 \text{ eV} \) for \( E^+ \), \( \chi_3 = 31 \text{ eV} \) for \( E^{2+} \), \( \chi_4 = 51 \text{ eV} \) for \( E^{3+} \);
- excitation energies that increase incrementally by 1 eV: \( \chi_{r,s} = s - 1 \text{ eV} \) in each stage;
- statistical weights \( g_{r,s} = 1 \) for all levels \((r, s)\).
partition function $U_r$:

$$U_r \equiv \sum g_{r,s} e^{-\chi_{r,s}/kT}$$

<table>
<thead>
<tr>
<th>$U_r$</th>
<th>5000 K</th>
<th>10000 K</th>
<th>20000 K</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U_1$</td>
<td>1.11</td>
<td>1.46</td>
<td>2.23</td>
</tr>
<tr>
<td>$U_2 = U_3 = U_4$</td>
<td>1.11</td>
<td>1.46</td>
<td>2.27</td>
</tr>
</tbody>
</table>

partition functions $U_r$ for Schadeenium E

Partition functions: of order unity and barely sensitive to temperature
Boltzmann distribution

\[ \frac{n_{r,s}}{N_r} = \frac{g_{r,s}}{U_r} e^{-\chi_{r,s}/kT} \]

The ground state always has the largest population (in “thermal” conditions, note for example “population inversion” in the case of masers)
Saha distribution

\[ \frac{N_{r+1}}{N_r} = \frac{1}{N_e} \frac{2U_{r+1}}{U_r} \left( \frac{2\pi m_e kT}{\hbar^2} \right)^{3/2} e^{-\chi_r/kT} \]

\[ e^{-\chi_r/kT} \text{ factor in Saha distribution Schadeenium} \]

\[ e^{-\chi_r/kT} \]

\[ T^{3/2} \text{ factor in the Saha distribution} \]

\[ T^{3/2} \times 10^3 \]

\[ \text{temperature [K]} \]

\[ r=1 : \chi_r = 7 \text{ eV} \]

\[ r=2 : \chi_r = 16 \text{ eV} \]

\[ r=3 : \chi_r = 31 \text{ eV} \]

\[ r=4 : \chi_r = 51 \text{ eV} \]
Saha distribution for Schadeenium E, $P_e = 1000$ dyne cm$^{-2}$

Only 2 ionisation stages significantly present
higher electron pressure $P_e$: “slower” ionisation
Saha-Boltzmann: Payne curves for Schadeenium

Schadeenium: ionization stages $r$ and excitation states $s$

population $n_{r,s} / N$

temperature [K]
Figure 6: The strengths of selected lines along the spectral sequence. Upper panel: variations of observed line strengths with spectral type in the Harvard sequence. The latter is plotted in reversed order on a non-linear scale that was obtained by making the peaks coincide with the corresponding peaks in the lower panel. The y-axis units are eye estimates on an arbitrary scale. Lower panel: Saha-Boltzmann predictions of the fractional concentration \( N_{r,s}/N \) of the lower level of the lines indicated in the upper panel, each labeled with its ionization stage, on logarithmic y-axis scales that are specified per species at the bottom, against temperature \( T \) along the x axis given in units of 1000 K along the top. The pressure was taken constant at \( P_e = N_e kT = 131 \text{ dyne cm}^{-2} = 13.1 \text{ Pascal} \). From Novotny (1973) who took it from Payne (1924).
Figure 5: A selection of stellar spectrograms illustrating the Harvard spectral sequence. These example
Payne's assumption:
spectral lines scale with the lower level population
emergent intensity from a thick medium (Rutten Sect. 3.7.3):

\[ I_\nu^+(\tau_\nu = 0, \mu = 1) = \int_0^\infty S_\nu(\tau_\nu) \, e^{-\tau_\nu} \, d\tau_\nu. \quad (3.20) \]

\[ \alpha^l_\nu = \frac{\pi e^2}{m_e c} n_l f_{lu} \varphi(\nu - \nu_0) \left[ 1 - e^{-h\nu_0/kT} \right]. \quad [\text{Rutten Sect. 5.4}] \]

The monochromatic line extinction per cm path length for a bound-bound transition between a lower level \( l \) and an upper level \( u \) is given by:

\[ \alpha^l_\lambda = \frac{\sqrt{\pi} e^2}{m_e c} \frac{\lambda^2}{c} b_l \frac{n_l^{\text{LTE}}}{N_E} N_H A_E f_{lu} \frac{H(a, v)}{\Delta \lambda_D} \left[ 1 - \frac{b_u}{b_l} e^{-hc/\lambda kT} \right], \quad (14) \]

[SSB]
Solar Flux atlas from 296 to 1300 nm

neutral Hydrogen (HI):

Ca II

Ca II
Hydrogen is much more abundant than Calcium. Why does the solar spectrum show such strong Ca II K line as compared to H-alpha?
Halpna is a $s=2 \rightarrow 3$ transition. The hydrogen $s=2$ population is of order $10^{-10}$ at solar temperatures. The Ca K transition has the ground state $s=1$ as lower level and has therefore the dominating population.

*also see question 4.26*
SSA 2.8 : Solar Ca II K versus Halpha

strength ratio: calcium abundance * Saha-Boltzmann ratio

\[ \frac{N(Ca)}{N(H)} \times \frac{N(Ca \text{ r}=2, s=1)}{N(H \text{ r}=1, s=2)} \]

For \( T=5000 \text{ K} \), the strength ratio is \( \sim 7600 \)

also see question 4.26
Halpha is more temperature sensitive in solar photosphere
here: Ca II s=1 population does not change much
→Ca K spectral line does not change much for different T
here: H I s=2 population does not change much
→Hα spectral line does not change much for different T
Hot stars: electrons from ionised Hydrogen
Cool stars: electrons from metals, “electron donors”
MK stellar classification system

<table>
<thead>
<tr>
<th>Class</th>
<th>Class characteristics</th>
<th>Type</th>
<th>Effective temperature ($T_{eff}$/K)</th>
</tr>
</thead>
<tbody>
<tr>
<td>O</td>
<td>hot stars with He II absorption lines; strong ultraviolet continuum</td>
<td>O5</td>
<td>40 000</td>
</tr>
<tr>
<td>B</td>
<td>He I lines attain maximum strength; no He II lines; H developing later</td>
<td>B0</td>
<td>28 000</td>
</tr>
<tr>
<td></td>
<td></td>
<td>B5</td>
<td>15 000</td>
</tr>
<tr>
<td>A</td>
<td>H lines attain maximum strength at A0, decreasing later; Ca II increasing</td>
<td>A0</td>
<td>9900</td>
</tr>
<tr>
<td></td>
<td></td>
<td>A5</td>
<td>8500</td>
</tr>
<tr>
<td>F</td>
<td>Ca II stronger; Fe and other metal lines appear</td>
<td>F0</td>
<td>6030</td>
</tr>
<tr>
<td></td>
<td></td>
<td>F5</td>
<td>6500</td>
</tr>
<tr>
<td>G</td>
<td>Ca II very strong; Fe and other metals strong; H weaker; solar-type spectrum</td>
<td>G0</td>
<td>6030</td>
</tr>
<tr>
<td></td>
<td></td>
<td>G5</td>
<td>5520</td>
</tr>
<tr>
<td>K</td>
<td>neutral metallic lines dominate; CH and CN bands developing; continuum weak in blue</td>
<td>K0</td>
<td>4900</td>
</tr>
<tr>
<td></td>
<td></td>
<td>K5</td>
<td>4130</td>
</tr>
<tr>
<td>M</td>
<td>very red; TiO$_2$ bands developing strongly</td>
<td>M0</td>
<td>3480</td>
</tr>
<tr>
<td></td>
<td></td>
<td>M5</td>
<td>2800</td>
</tr>
<tr>
<td></td>
<td></td>
<td>M8</td>
<td>2400</td>
</tr>
</tbody>
</table>

hot stars vs cool stars ($P_\nu = 100$ dyne cm$^{-2}$)

50% H ionization at $T = 9212$ K

neutral hydrogen fraction: $n(\text{H})$ vs temperature [K]
Hot stars have prominent hydrogen Balmer lines because of the long tail of H I population in the Saha distribution for high temperatures. Other elements are highly ionised (many multiple times). These ions have only few transitions in the optical.

### Stellar classification system

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<td>M8</td>
<td>2400</td>
</tr>
</tbody>
</table>
Planck function for $T = [5000 - 8000] \, \text{K}$
Planck function for $T = 10000\ \text{K}$

- Planck
- Wien, $B_\lambda \approx \frac{2hc^2}{\lambda^5} e^{-hc/\lambda kT}$
- Rayleigh-Jeans, $B_\lambda \approx \frac{2ckT}{\lambda^4}$
Planck function for $T = 10000$ K

- **Planck**

- **Wien**, $B_\lambda \approx \frac{2hc^2}{\lambda^5} e^{-hc/\lambda kT}$

- **Rayleigh-Jeans**, $B_\lambda \approx \frac{2ckT}{\lambda^4}$
Planck function for different temperatures

- $T = 1$ K
- $T = 10$ K
- $T = 100$ K
- $T = 1000$ K
- $T = 10000$ K

Wien displacement law

$B_\lambda$ [erg cm$^{-2}$ s$^{-1}$ cm$^{-1}$ ster$^{-1}$] vs. wavelength $\lambda$ [cm]
3.2 radiation through isothermal layer

\[ I_\lambda = I_\lambda(0) e^{-\tau} + B_\lambda \left(1 - e^{-\tau}\right) \]

Figure 3.3: Emergent intensity \( I_\nu(D) \) from a homogeneous medium against its optical thickness \( \tau_\nu(D) \). Optically thin, non-backlit objects produce \( I_\nu(D) = S_\nu \tau_\nu(D) = j_\nu D \) (lower curve, at left). If a background intensity \( I_\nu(0) \) illuminates the slab in the beam direction, there is enhancement of the intensity for \( I_\nu(0) < S_\nu \) (middle curve), reduction for \( I_\nu(0) > S_\nu \) (upper curve). For thick slabs with \( \tau_\nu(D) > 1 \), the emergent intensity \( I_\nu(D) \approx S_\nu \) independent of \( I_\nu(0) \).
$I_\lambda = I_\lambda(0) e^{-\tau} + B_\lambda (1 - e^{-\tau})$

Rutten Sect. 3.7.3:
$I_\nu \approx S_\nu \tau_\nu = B_\nu \tau_\nu$
Voigt profile

\[ V(a, u) \equiv \frac{1}{\Delta \lambda_D \sqrt{\pi}} \frac{a}{\pi} \int_{-\infty}^{+\infty} \frac{e^{-y^2}}{(u - y)^2 + a^2} \, dy. \]

\[ V(a, u) \approx \frac{1}{\Delta \lambda_D \sqrt{\pi}} \left[ e^{-u^2} + \frac{a}{\sqrt{\pi} u^2} \right] \]

Voigt profile, damping \( a = 0.01 \)

Voigt profile
Figure 12: The Schuster-Schwarzschild or reversing-layer model. The stellar surface radiates an intensity given by $B_\lambda(T_{\text{surface}})$. The shell around the surface only affects this radiation at the wavelengths where atoms provide a bound-bound transition between two discrete energy levels. These spectral line transitions cause attenuation $\tau_\lambda$. The layer has temperature $T_{\text{layer}}$ and gives a thermal contribution $B_\lambda(T_{\text{layer}}) [1 - \exp(-\tau_\lambda)]$ as in Eq. (11).
The Schuster-Schwarzschild or reversing-layer model. The stellar surface emits an intensity given by $B_\lambda(T_{\text{surface}})$. The shell around the surface only affects this radiation at the wavelengths where atoms provide a bound-bound transition between two discrete energy levels. These spectral line transitions cause attenuation $\tau_\lambda$. The layer has temperature $T_{\text{layer}}$ and gives a thermal contribution $B_\lambda(T_{\text{layer}}) \left[1 - \exp(-\tau_\lambda)\right]$ as in Eq. (11).
Equivalent Width

\[ W_\lambda \equiv \int \frac{I_{\text{cont}} - I(\lambda)}{I_{\text{cont}}} \, d\lambda \]

Figure 13: The equivalent width of a spectral line is the width of a rectangular piece of fully blocked spectrum with the same spectral area as the integrated line depression.
Equivalent Width

reversing layer: $a = 0.1$, $\tau_0 = 100$, equivalent width = 7.54

$T_{\text{surface}} = 5700$ K, $T_{\text{layer}} = 4200$ K
Equivalent Width

reversing layer: \( a = 0.1, \tau_0 = 100 \), equivalent width = 7.54

\( T_{\text{surface}} = 5700 \text{ K}, T_{\text{layer}} = 4200 \text{ K} \)

\[ I_\lambda \]

\( W_\lambda \)

Equivalent width spectral lines from reversing layer, \( a = 0.1 \)

For different values of \( \tau_0 \): 
- \( \tau_0 = 0.1 \), eq width = 0.14
- \( \tau_0 = 1.0 \), eq width = 1.12
- \( \tau_0 = 10.0 \), eq width = 3.43
- \( \tau_0 = 100.0 \), eq width = 7.54
- \( \tau_0 = 1000.0 \), eq width = 21.49

\( \text{optical thickness reversing layer } \tau_0 \) versus equivalent width.
Equivalent Width

Equivalent Width curve of growth

reversing layer: $a = 0.1$, $\tau_0 = 100$, equivalent width = 7.54

$T_{\text{surface}} = 5700$ K, $T_{\text{layer}} = 4200$ K

relative intensity $I_\lambda$

$-200 -150 -100 -50 0 50 100 150 200$

$u$: Doppler units

relative intensity $I_\lambda$

$-100 -50 0 50 100$

$u$: Doppler units

$\tau_0 = 0.1$, eq width = 0.14
$\tau_0 = 1.0$, eq width = 1.12
$\tau_0 = 10.0$, eq width = 3.43
$\tau_0 = 100.0$, eq width = 7.54
$\tau_0 = 1000.0$, eq width = 21.49

equivalent width spectral lines from reversing layer, $a = 0.1$
Equivalent Width

reversing layer: $a = 0.1$, $\tau_0 = 100$, equivalent width = 7.54

$T_{\text{surface}} = 5700$ K, $T_{\text{layer}} = 4200$ K

reversing layer: $a = 0.1$

equivalent width spectral lines from reversing layer

curve of growth

start of saturation
Figure 12: The Schuster-Schwarzschild or reversing-layer model. The stellar surface radiates an intensity given by $B_\lambda(T_{\text{surface}})$. The shell around the surface only affects this radiation at the wavelengths where atoms provide a bound-bound transition between two discrete energy levels. These spectral line transitions cause attenuation $\tau_\lambda$. The layer has temperature $T_{\text{layer}}$ and gives a thermal contribution $B_\lambda(T_{\text{layer}})[1 - \exp(-\tau_\lambda)]$ as in Eq. (11).
Midterm exam
Published Sep. 24, 2018 12:50 PM

The midterm exam will cover Chapters 1-5 in Rutten, problems Chapters 1-5, and the numerical exercise SSA (Monday 24 Sep we will discuss SSA during the lecture, pdf of the slides can be found here). During the exam, only the use of a calculator is permitted.

lectures week 38 (17 and 18 Sept): work on SSA
Published Sep. 14, 2018 2:53 PM
this week: discussion SSA and Rutten Chapter 5

next week:
Monday 1 Oct: questions mid-term
Tuesday 2 Oct: no lecture

midterm: Rutten Chapters 1-5 + questions, SSA